Graph Theory

Thomas R. Cameron

February 5, 2024

1 Key Topics

Today, we introduce the concept of forest and tree graphs. For further reading, see [1, Section 2.1] and [2, Section 2.1].

1.1 Forests and Trees

Recall that a cycle is a path that starts and ends at the same vertex. Let $G = (V, E)$ be a graph. If G contains no cycles, then we say that G is *acyclic*. Alternatively, we say that G is a *forest*. If G is a connected forest, then we say that G is a *tree*.

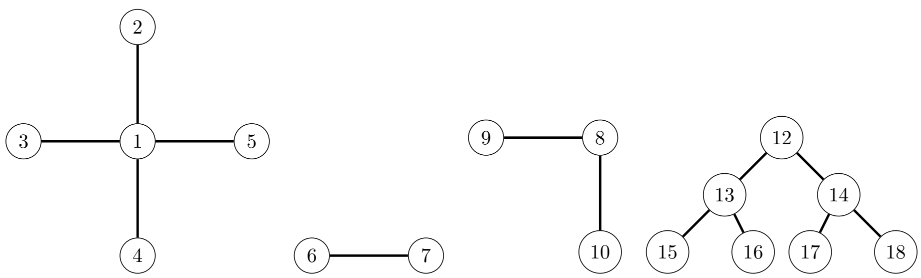

Figure 1: A forest made up of 4 trees

1.2 Properties of Trees

Theorem 1.1. *Let $G = (V, E)$. Then, G is a tree if and only if for all $u, v \in V$ there is a unique (u,v)-path.*

Proof. Suppose that G is a tree and let $u, v \in V$. Since G is connected, there exists a (u,v)-path. For the sake of contradiction, suppose there are two distinct paths P and Q. Let $\{x, y\}$ be the first edge in P that is not in Q and consider the graph $G - \{x, y\}$. Note that $G - \{x, y\}$ has an (x,y)-walk and therefore an (x,y)-path; indeed, an (x,y)-path can be constructed by taking P^{-1} from x to u, taking Q from u to v, and then taking P^{-1} from u to y. Let R denote an (x,y)-path in G that does not use the edge $\{x, y\}$. Then, adding the edge $\{x, y\}$ to R gives us a cycle in G, which contradicts G being a tree.

The converse is left for you to prove in Homework 3. □

Theorem 1.2. *Let $G = (V, E)$ be connected. Then, G is a tree if and only if every $e \in E$ is a cut edge.*

Proof. Let $e = \{u, v\}$ be an edge of G. By Theorem 1.1, there is only one path connecting u and v; in particular, $u \sim v$. Hence, if we delete the edge e there is no path connecting u and v, so $G - e$ is disconnected.

Conversely, suppose that every edge of G is a cut edge. By assumption, G is connected, so we only need to show that G is acyclic. For the sake of contradiction, suppose that there is a cycle in G. Let $e = \{x, y\}$ be an edge on this cycle and let P denote the (x,y)-path obtained from the cycle after removing the edge e. Since e is a cut edge, there exists two vertices $u, v \in V$ such that every (u,v)-path traverses the edge e. Let

Q denote a (u,v) path. Then, we can construct a (u,v)-walk in $G - e$ as follows: Use Q to go from u to x, then use P to go from x to y, then use Q to go from y to v. However, this implies that there is a (u,v)-path in $G - e$, which is a contradiction. □

2 Exercises

Draw all trees of order less than or equal to 5.

References

[1] D. Joyner, M. V. Nguyen, and D. Phillips, *Algorithmic Graph Theory and Sage*, 2013.

[2] K. Ruohonen, *Graph Theory*, 1st ed., 2013.