Graph Theory

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1 Key Topics

Today, we prove several properties regarding trees and their leafs, which allows us to prove further results regarding trees using mathematical induction. For further reading, see [1, Section 2.1] and [2, Section 2.2].

Recall that a tree is a connected acyclic graph and a forest is a graph whose connected components are trees.

1.1 Leafs

Let $G = (V, E)$ be a simple graph and let $v \in V$. We say that v is a leaf (pendent vertex) if $d(v) = 1$.

Proposition 1.1. *Let $G = (V, E)$ be a tree of order at least 2. Then, G has at least two leafs.*

Proof. Let l denote the length of a longest path in G and let

$$P = v_0 \sim v_1 \sim \ldots \sim v_l$$

denote a path of length l.

For the sake of contradiction, suppose that v_0 is not a leaf of G. Let $x \in N(v_0) \setminus \{v_1\}$ and note that x is not in the path P; otherwise, we would have a cycle in G. Hence,

$$x \sim v_0 \sim v_1 \sim \ldots \sim v_l$$

is a path in G that is longer than l.

A similar argument shows that v_l is a leaf of G. □

Proposition 1.2. *Let $G = (V, E)$ be a tree. If $v \in V$ be a leaf, then $G - v$ is a tree.*

Proof. Since G does not contain any cycles, it follows that $G-v$ does not contain any cycles. Let $x, y \in V \setminus \{v\}$. There is a unique (x,y)-path in G. Since v is a leaf, it follows that v cannot be in the (x,y)-path; hence, the (x,y) path exists in $G - v$. □

The converse of Proposition 1.2 is also true, we save the proof for Homework 3. We can use the result in Proposition 1.2 to construct trees of larger order from trees of smaller order. Moreover, we can use this result to develop inductive arguments for trees.

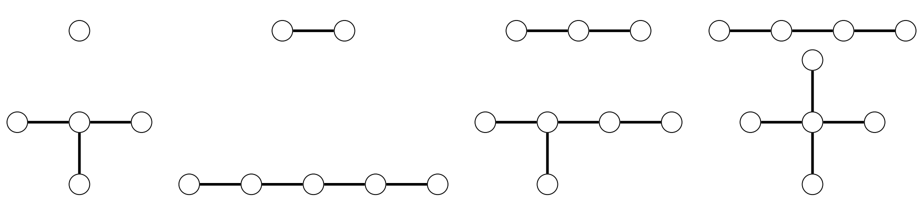

Figure 1: All non-isomorphic trees of order $n \leq 5$

1.2 Induction on Trees

Theorem 1.3. *Let $G = (V, E)$ be a tree with order $n \geq 1$. Then, $|E| = n - 1$.*

Proof. We proceed via induction on n. The base case, $n = 1$, is clear from Figure 1. Now, fix $n \geq 1$, and suppose that all trees of order n have $(n-1)$ edges. Let $G = (V, E)$ be a tree of order $(n+1)$. By Proposition 1.1, G has at least two leafs. Let $v \in V$ be a leaf of G. Then, by Proposition 1.2, $G - v$ is a tree of order n. Hence, by the induction hypothesis, $G - v$ has $(n - 1)$ edges. Since v is a leaf of G, it follows that G has n edges. □

1.3 Spanning Trees

Let $G = (V, E)$ be a simple graph. A *spanning subgraph* of G is any subgraph of G induced by $E' \subseteq E$. Moreover, a *spanning tree* of G is a spanning subgraph of G that is also a tree. For example, K_5 has several spanning trees as illustrated in

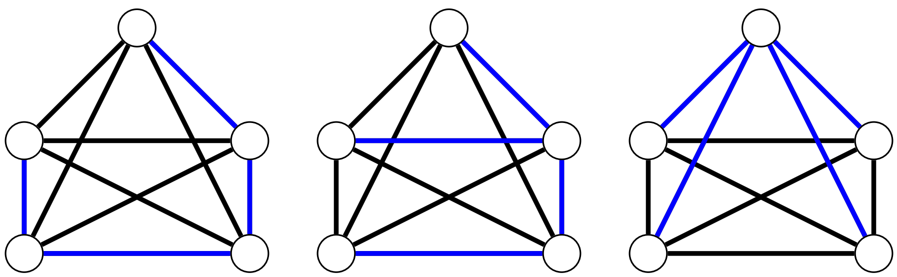

Figure 2: K_5 and all spanning subtrees

Theorem 1.4. *A graph has a spanning tree if and only if it is connected.*

Proof. Let $G = (V, E)$. Suppose that G has a spanning tree, which we denote by T. Let $u, v \in V(G) = V(T)$. Since T is a tree we know that T is connected. Hence, there exists a (u,v)-path in T. Since T is an induced subgraph of G, it follows that there is a (u,v)-path in G. Therefore, G is connected.

Conversely, suppose that G is connected. Let T be a connected spanning subgraph of G with the least number of edges. We claim that T is a spanning tree. By construction, T is connected. Furthermore, every $e \in E(T)$ is a cut edge; otherwise, $T - e$ would be a smaller connected spanning subgraph of G. Therefore, by Theorem 1.2 from 2/5/2024, T is a tree. □

Corollary 1.5. *Let $G = (V, E)$ be a connected graph of order $n \geq 1$. Then, G is a tree if and only if $|E| = n - 1$.*

2 Exercises

Prove the following result on trees using induction.

a. Every tree of order $n \geq 2$ has chromatic number 2.

b. Every tree of order $n \geq 2$ has independence number 2.

References

[1] D. Joyner, M. V. Nguyen, and D. Phillips, *Algorithmic Graph Theory and Sage*, 2013.

[2] K. Ruohonen, *Graph Theory*, 1st ed., 2013.