Graph Theory

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1 Key Topics

Today, we complete our proof that a connected graph of order n is a tree if and only if that graph has $(n-1)$ edges. Then, we introduce the concept of an Eulerian trail and circuit.

Recall the following results.

Theorem 1.1. *Let $G = (V, E)$ be a tree with order $n \geq 1$. Then, $|E| = n - 1$.*

Theorem 1.2. *A graph has a spanning tree if and only if it is connected.*

Now, we will prove our main result regarding the size of a tree.

Corollary 1.3. *Let $G = (V, E)$ be a connected graph of order $n \geq 1$. Then, G is a tree if and only if $|E| = n - 1$.*

Proof. The first direction follows from Theorem 1.1. For the converse, suppose that $|E| = n - 1$. Since G is connected, Theorem 1.2 implies that G has a spanning tree, which we denote by T. Now, we have

$$|E(T)| = |V(T)| - 1 = |V(G)| - 1 = |E(G)|.$$

Since $E(T) \subseteq E(G)$, it follows that $T = G$; hence, G is a tree. □

1.1 Eulerian Trails

Let $G = (V, E)$ be a simple graph and let W be a trail in G; recall, a trail is a walk that does not repeat an edge. If W uses every edge exactly once, then we say that W is an *Eulerian trail* If, in addition, the trail starts and ends at the same vertex, then we say that W is an *Eulerian circuit* (or *Eulerian tour*). Finally, if G has an Eulerian circuit, then we say that G is an *Eulerian graph.*

We now consider the question, which graphs are Eulerian? For example, see Figure 1.

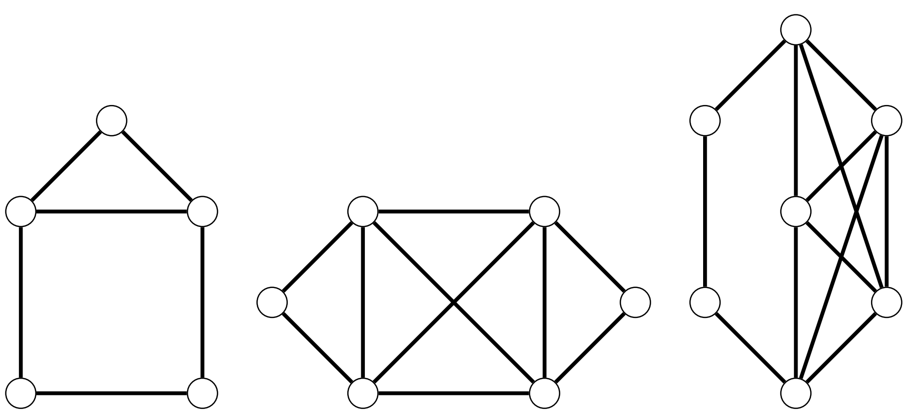

Figure 1: Which of these graphs are Eulerian?

It is often useful to start by considering necessary conditions. For instance:

- If G is Eulerian, then G has at most one non-trivial component.
- If G is Eulerian, then G does not contain any leafs.
- If G is Eulerian and of order $n \geq 3$, then G is not a tree.

The above necessary conditions imply that when characterizing Eulerian graphs we need only consider connected graphs; moreover, cycles will play an important role. Below is the main result.

Theorem 1.4. *Let $G = (V, E)$ be a connected graph. Then, the following statements are equivalent:*

a. G is Eulerian.

b. Every vertex of G has an even degree.

c. The edges of G can be partitioned into (edge-disjoint) cycles.

Proof.

$a \Rightarrow b$ Suppose that G is Eulerian and let W denote an Eulerian circuit in G. If $|V| = 1$, then the result is trivial since the only vertex of G has degree 0. Suppose that $|V| \geq 3$. Since G is connected, every vertex of G must be in W. Let $v \in V$ and let k denote the number of times v appears in W. Then, there are $2k$ edges incident on v that are traversed by W. Since W traverses every edge of G exactly once, it follows that $d(v) = 2k$.

$b \Rightarrow c$ Suppose that every vertex of G has an even degree. Again, the result is trivial if $|V| = 1$. Suppose that $|V| \geq 3$. Since G is not a tree, it follows that G has at least once cycle. We proceed via strong induction on the number of cycles in G, which we denote by k. The base case $k = 1$, is clear since G is C_n. Fix $k \geq 1$ and suppose that $b \Rightarrow c$ holds for all connected graphs with at most k cycles. Let G have $(k + 1)$ cycles. Let C denote one cycle of G and let G' denote the subgraph obtained from G by deleting all the edges of C. Since we are deleting the edges of a cycle, the degree of each vertex in that cycle is reduced by 2; hence, every vertex of G' has an even degree. Therefore, every vertex of G' has an even degree and the components of G' have no more than k cycles each. By the induction hypothesis, each component of G' can be partitioned into (edge-disjoint) cycles. This partition combined with C forms an edge-disjoint partition of cycles for G.

$c \Rightarrow a$ Suppose that the edges of G can be partitioned into disjoint cycles $C_1, \ldots, C_k$. Let C denote a circuit on G of maximum length such that

$$E(C) = E(C_{j_1}) \cup E(C_{j_2}) \cup \cdots \cup E(C_{j_m}),$$

for some collection of cycles $C_{j_1}, \ldots, C_{j_m}$. For the sake of contradiction, suppose there is an edge of G that is not in C. Then, since G is connected, there is an edge e that is not an edge of C and is incident with a vertex v in C. Furthermore, e must be an edge of a cycle C_i, for some i, where no edge of C_i is in C. Construct C' by patching C_i into C at the vertex v. Then, C' is a circuit of G with a larger length of C, which contradicts the maximality of C.

□