Graph Theory

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1 Key Topics

Today, we conclude the proof that a graph G is Eulerian if and only if every vertex of G has an even degree, which is true if and only if the edges of G can be partitioned into (edge-disjoint) cycles. Then, we introduce Hamiltonian graphs and discuss several fun problems on a graph.

1.1 Eulerian Graphs

Recall that a graph is Eulerian if it has a circuit that uses every edge exactly once.

Theorem 1.1. *Let $G = (V, E)$ be a connected graph. Then, the following statements are equivalent:*

a. G is Eulerian.

b. Every vertex of G has an even degree.

c. The edges of G can be partitioned into (edge-disjoint) cycles.

Proof.

$a \Rightarrow b$ Suppose that G is Eulerian and let W denote an Eulerian circuit in G. If $|V| = 1$, then the result is trivial since the only vertex of G has degree 0. Suppose that $|V| \geq 3$. Since G is connected, every vertex of G must be in W. Let $v \in V$ and let k denote the number of times v appears in W. Then, there are $2k$ edges incident on v that are traversed by W. Since W traverses every edge of G exactly once, it follows that $d(v) = 2k$.

$b \Rightarrow c$ Suppose that every vertex of G has an even degree. Again, the result is trivial if $|V| = 1$. Suppose that $|V| \geq 3$. Since G is not a tree, it follows that G has at least once cycle. We proceed via strong induction on the number of cycles in G, which we denote by k. The base case $k = 1$, is clear since G is C_n. Fix $k \geq 1$ and suppose that $b \Rightarrow c$ holds for all connected graphs with at most k cycles. Let G have $(k + 1)$ cycles. Let C denote one cycle of G and let G' denote the subgraph obtained from G by deleting all the edges of C. Since we are deleting the edges of a cycle, the degree of each vertex in that cycle is reduced by 2; hence, every vertex of G' has an even degree. Therefore, every vertex of G' has an even degree and the components of G' have no more than k cycles each. By the induction hypothesis, each component of G' can be partitioned into (edge-disjoint) cycles. This partition combined with C forms an edge-disjoint partition of cycles for G.

$c \Rightarrow a$ Suppose that the edges of G can be partitioned into disjoint cycles $C_1, \ldots, C_k$. Let C denote a circuit on G of maximum length such that

$$E(C) = E(C_{j_1}) \cup E(C_{j_2}) \cup \cdots \cup E(C_{j_m}),$$

for some collection of cycles $C_{j_1}, \ldots, C_{j_m}$. For the sake of contradiction, suppose there is an edge of G that is not in C. Then, since G is connected, there is an edge e that is not an edge of C and is incident with a vertex v in C. Furthermore, e must be an edge of a cycle C_i, for some i, where no edge of C_i is in C. Construct C' by patching C_i into C at the vertex v. Then, C' is a circuit of G with a larger length of C, which contradicts the maximality of C.

□

1.2 Hamiltonian Graphs

Recall that a graph is Hamiltonian if it has a cycle that uses every vertex exactly once. It is important to note that not every Eulerian graph is Hamiltonian and not every Hamiltonian graph is Eulerian. For example, consider the graphs in Figure 1.

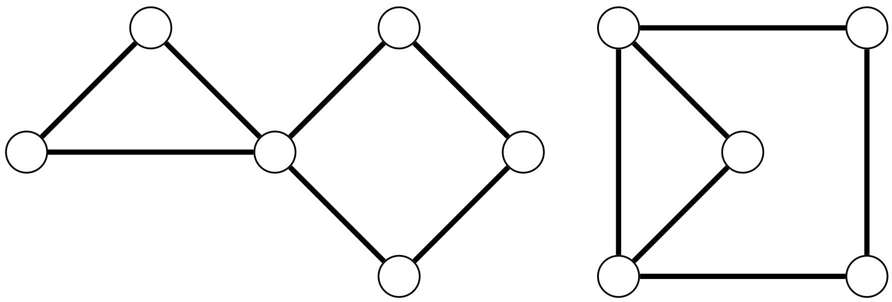

Figure 1: An Eulerian graph that is not Hamiltonian and a Hamiltonian graph that is not Eulerian

2 Exercises

I. Let $G = (V, E)$ be a graph where $|V| = 64$ and each vertex corresponds to a square on a chessboard. Let $\{u, v\} \in E$ if and only if a knight can move from square u to square v on a chessboard. Is G Eulerian? Is G Hamiltonian?

II.