Graph Theory

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1 Key Topics

Today, we begin our discussion of planar graphs.

1.1 Planar Graphs

A graph G is *planar* if it can be drawn in the plane in such a way that pairs of edges intersect only at vertices, if at all. For instance, every tree is planar and every cycle is planar. If G has no such representation, then we say it is *nonplanar*. As we will see, $K_{3,3}$ is a nonplanar graph. A drawing of a planar graph in which edges intersect only at vertices, if at all, is called a *planar representation*. In Figure 1 we see a planar graph drawn three different ways, note that the last drawing is not a planar representation.

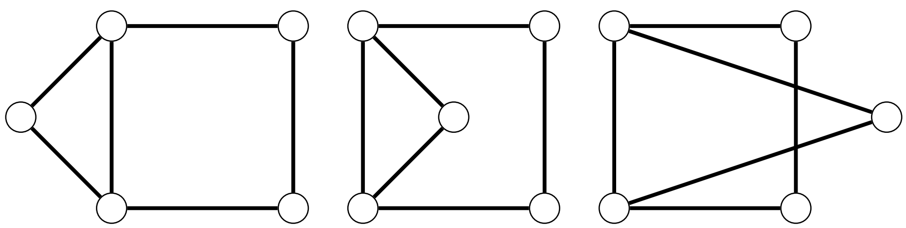

Figure 1: A planar graph drawn three different ways

Given a planar representation of a graph G, a *region* is a maximal section of the plane in which any two points can be joined by a curve that does not intersect any part of G. For example, in Figure 2, we see a planar graph and its three regions.

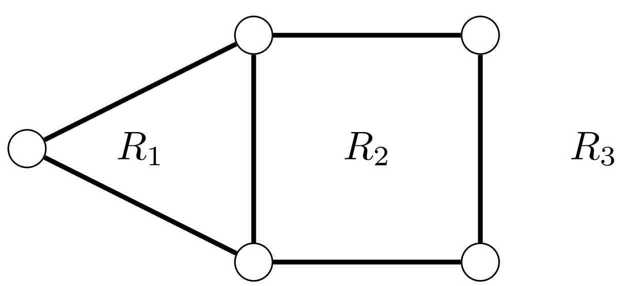

Figure 2: A planar graph and its regions

We can think of a region as being bounded by edges. A single edge can come into contact with either one or two regions. We say that an edge e *bounds* a region R if e comes into contact with R and another region different from R. We define the *bound degree* of a region R, denoted $b(R)$, by the number of edges that bound R. For example, in Figure 2, we have $b(R_1) = 3$, $b(R_2) = 4$, and $b(R_3) = 5$.

2 Exercises

I. Prove that all trees are planar.

II. Prove that all Eulerian graphs are planar.

III. For each graph below determine the number of vertices, edges, and regions.