Graph Theory

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1 Key Topics

Today, we continue our discussion of planar graphs and prove Euler's formula.

Recall that a graph G is *planar* if it can be drawn in the plane in such a way that pairs of edges intersect only at vertices, if it all. A drawing of a plane graph in which edges intersect only at vertices, if at all, is called a planar representation. Given a planar representation of G, a region (face) is a maximal section of the plane in which any two points can be joined by a curve that does not intersect any part of G. For example, in Figure 1, we see a planar graph and its three regions.

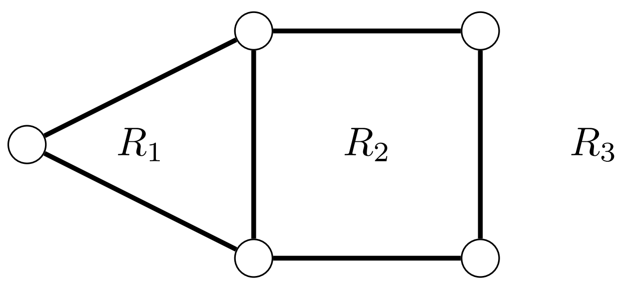

Figure 1: A planar graph and its regions

We say that an edge e bounds a region R if e comes into contact with R and another region different from R. Furthermore, the bound degree of a region R, denoted $b(R)$, is equal to the number of edges that bound R.

Note that, in Figure 1, we have $b(R_1) = 3$, $b(R_2) = 4$, and $b(R_3) = 5$. Moreover, the number of vertices n, number of edges m, and the number of faces f, satisfy the following formula, known as Euler's formula:

$$n - m + f = 2. \tag{1}$$

1.1 Euler's Formula

Theorem 1.1. *If G is a connected planar graph with n vertices, m edges, and f faces. Then, $n-m+f = 2$.*

Proof. We proceed via induction on m. If $m = 0$, then G must be a graph with 1 vertex and 1 face. Hence for any planar representation of G, the number of faces satisfies $f = 1$, so $n - m + f = 1 - 0 + 1 = 2$. Now, let $m \geq 1$ and suppose Euler's formula holds for all connected planar graphs with less than m edges.

Let G be a connected planar graph with m edges and consider two cases: one where G is a tree and the other where G is not a tree. If G is a tree, then $m = n-1$ and $f = 1$; hence, $n-m+f = n-(n-1)+1 = 2$.

Now, suppose that G is not a tree and let C denote a cycle in G. Let e be an edge of C and note that e must bound a region R (the interior of C) and e is not a cut edge. Therefore, $G - e$ is a connected Eulerian graph with one fewer regions than G, one fewer edges, and the same number of vertices. Since the induction hypothesis applies to $G - e$, we have

$$n - (e - 1) + (f - 1) = 2,$$

which implies that $n - e + f = 2$. □

Euler's formula is useful for identifying non-planar graphs and other sufficient conditions for planar graphs.

Corollary 1.2. *If G is a connected planar graph with $n \geq 3$ vertices and m edges, then $m \leq 3n - 6$. Furthermore, if equality holds then every region has bound degree 3.*

Proof. Consider the sum

$$S = \sum_R b(R).$$

Note that every boundary edge is a boundary edge for two regions; hence, $S \leq 2m$. Further, since each region is bounded by at least 3 edges, we have $S \geq 3f$. Therefore,

$$3f \leq 2m \;\Rightarrow\; 3(2 + m - n) \leq 2m \;\Rightarrow\; m \leq 3n - 6.$$

□

Corollary 1.3. *If G is a connected planar graph, then $\delta(G) \leq 5$.*

Proof. If $n \geq 6$, then the result is obvious. Suppose that $n > 6$ and note that

$$\sum_{v \in V} d(v) = 2m \leq 2(3n - 6) = 6n - 12.$$

If each vertex has degree 6 or more, then $\sum_{v \in V} d(v) \geq 6n$. □

1.2 Nonplanar Graphs

Proposition 1.4. *$K_{3,3}$ is non-planar.*

Proposition 1.5. *K_5 is non-planar.*

2 Exercises

I. Prove Proposition 1.4

II. Prove Proposition 1.5