Graph Theory

Thomas R. Cameron

February 19, 2024

1 Key Topics

Today, we continue our discussion of planar graphs and discuss Kuratowksi's Theorem which provides a complete description of all planar graphs.

Recall that a graph G is *planar* if it can be drawn in the plane in such a way that pairs of edges intersect only at vertices, if it all. A drawing of a plane graph in which edges intersect only at vertices, if at all, is called a planar representation. Given a planar representation of G, a region (face) is a maximal section of the plane in which any two points can be joined by a curve that does not intersect any part of G. For example, in Figure 1, we see a planar graph and its six regions.

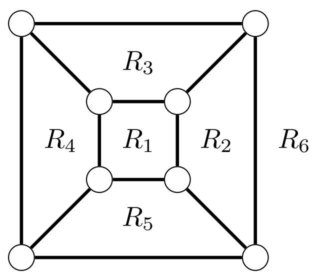

Figure 1: A planar graph and its regions

Last time, we proved that Euler's formula ($n - m + f = 2$) holds for all connected planar graphs. In Figure 1, we have $n = 8$, $m = 12$, and $f = 6$; hence, $8 - 12 + 6 = 2$. Then, we proved that $K_{3,3}$ is non-planar by contradiction using Euler's formula. Moreover, we proved that if G is connected with $n \geq 3$, then $m \leq 3n - 6$. Finally, we used this result to prove that K_5 is non-planar.

1.1 Subdivisions and Minors

Let $G = (V, E)$ be a simple graph. A subdivision of G is formed by replacing an edge $e \in E$ by a path (possibly K_2). For example, the graph in Figure 2 below is a subdivision of K_5.

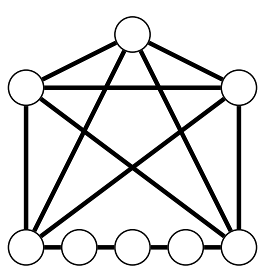

Figure 2: A subdivision of K_5

It is worth noting the graph in Figure 2 has a K_5 minor. Indeed every subdivision of a graph G has G as a minor. However, the converse is not true, i.e., there are graphs that have G as a minor that are not subdivisions of G.

Subdivisions play an important role in the characterization of planar graphs, as eluded to by the following proposition.

Proposition 1.1. *A graph is planar if and only if all of its subdivisions are planar.*

The following result is a famous characterization of planar graphs due to Kazimierz Kuratowski in 1930.

Theorem 1.2 (Kuratowski)**.** *A graph G is planar if and only if it contains no subdivision of $K_{3,3}$ or K_5.*

Kuratowksi's theorem can be restated in terms of minors, as was done by Klaus Wagner in 1937. Recall that a minor of a graph is a graph obtain from zero or more edge contractions, edge deletions, or vertex deletions.

Proposition 1.3. *A graph is planar if and only if all of its minors are planar.*

Theorem 1.4 (Wagner)**.** *A graph G is planar if and only if its minors don't include $K_{3,3}$ or K_5.*

Wagner's theorem is an example of a forbidden minor characterization. The Robertson-Seymour theorem states that undirected graphs, partially ordered by the graph minor relationship, form a well-quasi-ordering. The Robertson-Seymour theorem was proved by Neil Robertson and Paul D. Seymour in a series of twenty papers spanning over 500 pages from 1983 to 2004. This famous theorem implies that every property of graphs that is preserved by deletions and edge contractions has an analogous forbidden minor characterization. For example, consider the following characterizations.

Theorem 1.5. *A connected graph G is a tree if and only if G does not contain a C_3 minor.*

Theorem 1.6. *A graph G is complete bipartite if and only if G does not contain a C_{2k+1} minor.*

2 Exercises

I. Prove Proposition 1.1.

II. Prove Proposition 1.3.

III. Use Theorem 1.4 to prove that the Peterson graph is not planar.

IV. Show that the Peterson graph contains a $K_{3,3}$ subdivision.