Graph Theory

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1 Key Topics

Today, we begin our discussion of matrices associated with a graph.

1.1 The Symmetric Matrices of a Graph

Let $G = (V, E)$ be a simple graph with vertex set $V = \{v_1, \ldots, v_n\}$. Also, let $S_n(\mathbb{R})$ denote the set of $n \times n$ real symmetric matrices. For each $A \in S_n(\mathbb{R})$ we denote its (i, j) entry a_{ij}. Since A is symmetric, it follows that $a_{ij} = a_{ji}$ for all $i \neq j$. Moreover, the set of matrices A associated with the graph G is defined by

$$\mathcal{S}(G) = \{A \in S_n(\mathbb{R})\colon \forall i \neq j,\ a_{ij} \neq 0 \Leftrightarrow \{v_i, v_j\} \in E\}.$$

For example,

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \in \mathcal{S}(P_4), \quad \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \notin \mathcal{S}(P_4).$$

Note that there is no restriction on the diagonal entries of $A \in \mathcal{S}(G)$ since G is a simple graph with no loops.

1.2 Permutations and Isomorphisms

Consider the labeled graph G and an isomorphic graph G' in Figure 1, where the corresponding isomorphism is given by

$$f = \{(1, 4'), (2, 3'), (3, 2'), (4, 1'), (5, 5')\}.$$

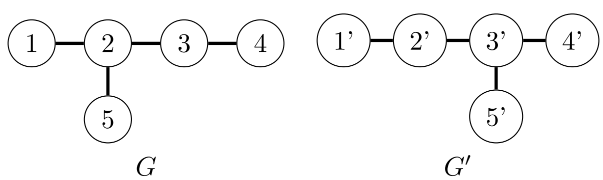

Figure 1: A graph and an isomorhpic graph

In general, the graph G and G' from Figure 1 have the following symmetric matrix form

$$A = \begin{bmatrix} a_1 & a_{12} & 0 & 0 & 0 \\ a_{12} & a_2 & a_{23} & 0 & a_{25} \\ 0 & a_{23} & a_3 & a_{34} & 0 \\ 0 & 0 & a_{34} & a_4 & 0 \\ 0 & a_{52} & 0 & 0 & a_5 \end{bmatrix}, \quad A' = \begin{bmatrix} a_1' & a_{12}' & 0 & 0 & 0 \\ a_{12}' & a_2' & a_{23}' & 0 & 0 \\ 0 & a_{23}' & a_3' & a_{34}' & a_{35}' \\ 0 & 0 & a_{34}' & a_4' & 0 \\ 0 & 0 & a_{35}' & 0 & a_5' \end{bmatrix}$$

Since G and G' are isomorphic graphs, there should be a relationship between the matrices A and A'. Indeed, their zero non-zero patterns are permutation similar. In particular, there exists a permutation matrix P such that for every $A' \in \mathcal{S}(G')$ there is a $A \in \mathcal{S}(G)$ such that $PAP^{-1} = A'$.

A permutation matrix P is a square binary matrix (entries are either 0 or 1) such that there is exactly one entry 1 in each row and column. By definition, a permutation matrix is orthonormal (its columns are orthogonal and have unit length). Hence, the inverse of a permutation matrix is its transpose, i.e., $P^{-1} = P^T$.

We can identify the specific permutation in the similarity transformation between A and A' by using the isomorphism f. In particular, for each ordered pair $(i, j) \in f$, we mark the (j, i) entry of P equal to 1; all other entries of P are set equal to 0. Since f is a bijection, it follows that P is a permutation matrix. In this particular example, we have

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Note that

$$\begin{aligned} PAP^{-1} &= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 & a_{12} & 0 & 0 & 0 \\ a_{12} & a_2 & a_{23} & 0 & a_{25} \\ 0 & a_{23} & a_3 & a_{34} & 0 \\ 0 & 0 & a_{34} & a_4 & 0 \\ 0 & a_{52} & 0 & 0 & a_5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & a_{12} & a_1 & 0 \\ 0 & a_{23} & a_2 & a_{12} & a_{25} \\ a_{34} & a_3 & a_{23} & 0 & 0 \\ a_4 & a_{34} & 0 & 0 & 0 \\ 0 & 0 & a_{25} & 0 & a_5 \end{bmatrix} \\ &= \begin{bmatrix} a_4 & a_{34} & 0 & 0 & 0 \\ a_{34} & a_3 & a_{23} & 0 & 0 \\ 0 & a_{23} & a_2 & a_{12} & a_{25} \\ 0 & 0 & a_{12} & a_1 & 0 \\ 0 & 0 & a_{25} & 0 & a_5 \end{bmatrix} \end{aligned}$$

2 Exercises

I. Find the general form of the symmetric matrix associated with

- The empty graph E_n,
- The path graph P_n,
- The cycle graph C_n,
- The star graph S_n,
- The complete graph K_n.

II. Let Q be an orthonormal matrix. Show that $Q^T Q = I$. Hence, $Q^T = Q^{-1}$.