Graph Theory

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1 Key Topics

Today, we continue our discussion of matrices associated with a graph. Further, introduce the adjacency and Laplacian matrices and discuss some of their properties.

1.1 Permutation Similarity

Consider the labeled graph G and an isomorphic graph G' in Figure 1, where the corresponding isomorphism is given by

$$f = \{(1, 1'), (2, 4'), (3, 2'), (4, 5'), (5, 3')\}$$

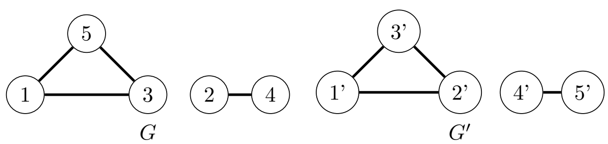

Figure 1: A graph and an isomorhpic graph

Then, there exists a permutation matrix P such that for each $A \in \mathcal{S}(G)$, $PAP^{-1} \in \mathcal{S}(G')$. Indeed, we can construct P from the isomorphism f; in particular, for each $(i, j) \in f$ we mark the (j, i) entry of P equal to 1; all other entries of P are set equal to 0. Since f is a bijection, it follows that P is a permutation matrix (why?).

To see that $PAP^{-1} \in \mathcal{S}(G)$, let $A' = PAP^{-1} = PAP^T$. Then, $P^T A' P = A$. Furthermore, the (i, j) entry of A can be denoted by

$$\mathbf{e}_i^T A \mathbf{e}_j = \mathbf{e}_i^T P^T A' P \mathbf{e}_j,$$

where $\mathbf{e}_i$ denotes the ith standard basis vector, i.e., all entries of $\mathbf{e}_i$ are zero except for the single one entry in the ith coordinate. The remainder of this exercise is left to you in Homework 5.

For the graph in Figure 1 note that

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \text{ and } PAP^T = \begin{bmatrix} d_1 & a_{13} & a_{15} & 0 & 0 \\ a_{13} & d_3 & a_{35} & 0 & 0 \\ a_{15} & a_{35} & d_5 & 0 & 0 \\ 0 & 0 & 0 & d_2 & a_{24} \\ 0 & 0 & 0 & a_{24} & d_4 \end{bmatrix}.$$

In particular, note that the connected components appear in block diagonal form of the matrix.

1.2 Laplacian and Adjacency Matrices

Let $G = (V, E)$ be a simple graph with vertex set $V = \{1, 2, \ldots, n\}$. The *adjacency matrix* $A \in \mathcal{S}(G)$ is a binary matrix where $a_{ij} = 1$ if and only if $\{i, j\} \in E$. Let D be a diagonal matrix such that $d_{ii} = d(i)$, where $d(i)$ denotes the degree of vertex i. Then, the *Laplacian matrix* is defined by $L = D - A$.

The Laplacian matrix has several important properties, which we state below as propositions.

Proposition 1.1. *The trace (sum of the diagonal entries) of the Laplacian satisfies*

$$\operatorname{tr}(L) = \sum_{i \in V} d(i) = 2\,|E|$$

Proposition 1.2. *Let* $\mathbf{e}$ *denote the all ones vector. Then,* $L\mathbf{e} = 0$*.*

From Proposition 1.2, it follows that the Laplacian matrix is not invertible and has zero determinant. In fact, we can generalize Proposition 1.2 to graphs with k connected components. Note that if G has k connected components, then (perhaps after re-labeling of the vertices), the Laplacian matrix can be written as

$$L = \begin{bmatrix} L_1 & & & \\ & L_2 & & \\ & & \ddots & \\ & & & L_k \end{bmatrix},$$

where L_i denotes the Laplacian of the connected component $G[V_i]$, i.e., the induced subgraph of the ith equivalence class under the connected-to relation. For each i define the incidence vector

$$\mathbf{x}_i[u] = \begin{cases} 1 & u \in V_i \\ 0 & u \notin V_i \end{cases}$$

Proposition 1.3. *Let* G *have* k *connected components. Then, for* $i = 1, \ldots, k$*, the* i*th incidence vector satisfies* $L\mathbf{x}_i = 0$*.*

2 Exercises

I. Find the Laplacian of the empty, complete, cycle, path, and star graphs.

II. Prove Proposition 1.1

III. Prove Proposition 1.2

IV. Prove Proposition 1.3