Graph Theory

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1 Key Topics

Today, we review eigenvalues and eigenvectors and being our investigation of the spectral properties of matrices associated with graphs.

1.1 Invertibility, Rank, and Nullity

Let $A \in \mathbb{C}^{n\times n}$ denote an $n \times n$ complex matrix. The matrix A is *invertible* if the matrix equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for each $\mathbf{b} \in \mathbb{C}^n$. Note that A being invertible is equivalent to its echelon form is upper triangular with non-zero diagonal entries, which is equivalent to $\det(A) \neq 0$. If the matrix A is invertible, we define the *inverse* of A as the matrix A^{-1} whose ith column is the unique solution of $A\mathbf{x} = \mathbf{e}_i$, where $\mathbf{e}_i$ denotes the ith standard basis vector. The inverse of A satisfies $AA^{-1} = A^{-1}A = I$.

If A is not invertible, then we say it is *singular*. In this case, there is more than one solution to the matrix equation $A\mathbf{x} = 0$; in particular, there exists a non-zero $\mathbf{x} \in \mathbb{C}^n$ such that $A\mathbf{x} = 0$. We define the *null space* of A as $\mathcal{N}(A) = \{\mathbf{x} \colon A\mathbf{x} = 0\}$. The dimension of the null space is known as the *nullity* of A, denoted $\text{nullity}\,(A)$.

If there exists a non-zero $\mathbf{x} \in \mathbb{C}^n$ such that $A\mathbf{x} = 0$, then the columns of A are linearly dependent. Hence, the number of linearly independent columns is less than n. The *rank* of A, denoted $\text{rank}\,(A)$, is the maximum number of linearly independent columns of A. The famous rank-nullity theorem states that $\text{rank}\,(A) + \text{nullity}\,(A) = n$.

1.2 Eigenvalues and Eigenvectors

An *eigenvalue* of A is any scalar $\lambda \in \mathbb{C}$ such that $\lambda I - A$ is singular, i.e., there exists a non-zero $\mathbf{x} \in \mathbb{C}^n$ such that $(\lambda I - A)\,\mathbf{x} = 0$. The vector $\mathbf{x}$ is known as an *eigenvector* of A corresponding to λ. Moreover, note that $A\mathbf{x} = \lambda\mathbf{x}$.

To identify eigenvalues of A, we consider the *characteristic polynomial* of A, which is defined as follows

$$p_A(\lambda) = \det(\lambda I - A).$$

Since $\lambda I - A$ is singular when $\det(\lambda I - A) = 0$, it follows that the roots of $p_A(\lambda)$ are the eigenvalues of A.

Example 1.1. Let

$$A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}.$$

Note that

$$\begin{aligned} p_A(\lambda) = \det(\lambda I - A) &= \det\left(\begin{bmatrix} \lambda - 2 & -3 \\ -3 & \lambda + 6 \end{bmatrix}\right) \\ &= (\lambda - 2)(\lambda + 6) - 9 \\ &= \lambda^2 + 4\lambda - 21 = (\lambda + 7)(\lambda - 3). \end{aligned}$$

The eigenvalues are the roots of the characteristic polynomial; thus, we have

$$\lambda_1 = -7, \ \lambda_2 = 3.$$

We can determine corresponding eigenvectors by investigating the null spaces

$$\mathcal{N}(\lambda_1 I - A) = \mathcal{N}\left(\begin{bmatrix} -9 & -3 \\ -3 & -1 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 1 \\ -3 \end{bmatrix}\right).$$

and

$$\mathcal{N}(\lambda_2 I - A) = \mathcal{N}\left(\begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right).$$

Therefore, corresponding eigenvectors can be chosen as

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

□

Theorem 1.2. *Let $A \in \mathbb{C}^{n \times n}$ and let $S \in \mathbb{C}^{n \times n}$ be invertible. Then, the matrix $B = SAS^{-1}$ has the same eigenvalues as the matrix A.*

Proof. Let $\lambda \in \mathbb{C}$ be an eigenvalue of A with corresponding eigenvector $\mathbf{x}$. Then, $A\mathbf{x} = \lambda\mathbf{x}$ and

$$BS\mathbf{x} = SA\mathbf{x} = \lambda S\mathbf{x}.$$

Since S is invertible and $\mathbf{x}$ is non-zero, it follows that $S\mathbf{x}$ is non-zero. Hence, λ is an eigenvalue of B with corresponding eigenvector $S\mathbf{x}$. □

1.3 Symmetric Matrices

In Example 1.1, we saw that the 2×2 real symmetric matrix had 2 real eigenvalues. Moreover, we saw that the eigenvectors corresponding to distinct eigenvalues where orthogonal. These observations are not coincidental; in fact, they are true of all symmetric matrices. Below we state (without proof) the spectral theorem for symmetric matrices.

Theorem 1.3. *Let $A \in \mathbb{R}^{n \times n}$ be symmetric. Then,*

a. A has n real eigenvalues, counting multiplicities.

b. The maximum number of linearly independent eigenvectors corresponding to each eigenvalue λ is equal to the multiplicity of λ is a root of the characteristic polynomial of A.

c. The eigenvectors of A corresponding to distinct eigenvalues are orthogonal.

d. The matrix A is orthogonally diagonalizable.

2 Exercises

I. Find the eigenvalues and corresponding eigenvectors of the Laplacian matrix of K_3.