Graph Theory

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1 Key Topics

Today, we continue our review of the eigenvalues and eigenvectors of a matrix. Further, we introduce the spectral theorem for symmetric matrices and the notion of an M-matrix.

Recall that an eigenvalue of the matrix $A \in \mathbb{C}^{n\times n}$ is a scalar $\lambda \in \mathbb{C}$ such that $\lambda I - A$ is singular, i.e., there exists a non-zero $\mathbf{x} \in \mathbb{C}^n$ such that $(\lambda I - A)\,\mathbf{x} = 0$. The vector $\mathbf{x}$ is an eigenvector of A corresponding to λ. Moreover, note that $A\mathbf{x} = \lambda\mathbf{x}$.

Example 1.1. Consider the Laplacian matrix associated with K_3:

$$L = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & -2 \end{bmatrix} = 3I - \mathbf{e}\mathbf{e}^T,$$

where I denotes the 3×3 identity matrix and $\mathbf{e}$ denotes the all ones vector of dimension 3.

Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix},\ \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}.$$

Then one can readily verify that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are eigenvectors of L corresponding to the eigenvalues $\lambda_1 = 0$, $\lambda_2 = 3$, and $\lambda_3 = 3$.

A major theme over the next few weeks will be to investigate to what extent the eigenvalues (spectra) of the matrices in $\mathcal{S}(G)$ characterize the properties of $\mathcal{G}$. Since the properties of $\mathcal{G}$ are not influenced by the re-labeling of vertices, it is vital the same be true of the eigenvalues of $\mathcal{S}(G)$. Fortunately, we have the following result.

Theorem 1.2. *Let $A \in \mathbb{C}^{n\times n}$ and let $S \in \mathbb{C}^{n\times n}$ be invertible. Then, the matrix $B = SAS^{-1}$ has the same eigenvalues as the matrix A.*

Proof. Let $\lambda \in \mathbb{C}$ be an eigenvalue of A with corresponding eigenvector $\mathbf{x}$. Then, $A\mathbf{x} = \lambda\mathbf{x}$ and

$$BS\mathbf{x} = SA\mathbf{x} = \lambda S\mathbf{x}.$$

Since S is invertible and $\mathbf{x}$ is non-zero, it follows that $S\mathbf{x}$ is non-zero. Hence, λ is an eigenvalue of B with corresponding eigenvector $S\mathbf{x}$. □

1.1 Symmetric Matrices

In Example 1.1, we saw that the 3×3 real symmetric matrix had 3 real eigenvalues. Moreover, we saw that the eigenvectors corresponding to distinct eigenvalues where orthogonal. These observations are not coincidental; in fact, they are true of all symmetric matrices. Below we state (without proof) the spectral theorem for symmetric matrices.

Theorem 1.3. *Let $A \in \mathbb{R}^{n\times n}$ be symmetric. Then,*

a. A has n real eigenvalues, counting multiplicities.

b. The maximum number of linearly independent eigenvectors corresponding to each eigenvalue λ is equal to the multiplicity of λ is a root of the characteristic polynomial of A.

c. The eigenvectors of A corresponding to distinct eigenvalues are orthogonal.

d. The matrix A is orthogonally diagonalizable.

1.2 M-Matrices

To understand the definition of an M-matrix, we must first introduce the non-negative matrices and the spectral radius of a matrix. A *non-negative* matrix A, denoted $A \geq 0$, is a real matrix whose entries are non-negative. The *spectral radius* of a matrix A, denote $\rho(A)$, is the maximum magnitude of an eigenvalue of A, i.e.,

$$\rho(A) = \{|\lambda| : \ \det(\lambda I - A) = 0\}\,.$$

Now, a real matrix A is an *M-matrix* if it can be written in the following form:

$$A = sI - B,$$

where $s \geq \rho(B)$ and $B \geq 0$.

We can bound the spectral radius of a matrix using Gerschgorin's theorem.

Theorem 1.4. *Let $A \in \mathbb{C}^{n\times n}$. Then for any eigenvalue λ of A there exists an integer $k \in \{1, \dots, n\}$ such that*

$$|\lambda - a_{k,k}| \leq \sum_{j=1, j\neq k}^{n} |a_{k,j}|\,.$$

Proof. Let λ be an eigenvalue of A and let $\mathbf{x}$ be a corresponding eigenvector. Then, for each $i \in \{1, \dots, n\}$, we have

$$\sum_{j=1}^{n} a_{i,j} x_j = \lambda x_i$$

Since $\mathbf{x}$ is non-zero, there is an integer k such that $0 < |x_k| = \max\{|x_i| : i \in \{1, \dots, n\}\}$. For this k, we have

$$(\lambda - a_{k,k}) x_k = \sum_{j=1, j\neq k} a_{k,j} x_j$$

Taking absolute values and applying the triangle inequality:

$$|\lambda - a_{k,k}|\,|x_k| \leq \sum_{j=1, j\neq k} |a_{k,j}|\,|x_j| \leq \sum_{j=1, j\neq k} |a_{k,j}|\,|x_k|\,.$$

Therefore,

$$|\lambda - a_{k,k}| \leq \sum_{j=1, j\neq k}^{n} |a_{k,j}|\,.$$

□

2 Exercises

I. Show that the Laplacian matrix corresponding to each graph of order 3 is an M-matrix.