Graph Theory

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March 13, 2024

1 Key Topics

Today, we continue our investigation of eigenvalues, eigenvectors, and M-matrices.

Recall that an eigenvalue of the matrix $A \in \mathbb{C}^{n\times n}$ is a scalar $\lambda \in \mathbb{C}$ such that $\lambda I - A$ is singular, i.e., there exists a non-zero $\mathbf{x} \in \mathbb{C}^n$ such that $(\lambda I - A)\,\mathbf{x} = 0$. The vector $\mathbf{x}$ is an eigenvector of A corresponding to λ. Moreover, note that $A\mathbf{x} = \lambda\mathbf{x}$.

Also, recall that $A \in \mathbb{R}^{n\times n}$ is an M-matrix if it can be written in the following form:

$$A = sI - B,$$

where $s \geq \rho(B)$ and $B \geq 0$. We can bound the spectral radius of a matrix using Gerschgorin's theorem.

Theorem 1.1. *Let $A \in \mathbb{C}^{n\times n}$. Then for any eigenvalue λ of A there exists an integer $k \in \{1,\ldots,n\}$ such that*

$$|\lambda - a_{k,k}| \leq \sum_{j=1,j\neq k}^{n} |a_{k,j}|\,.$$

Proof. Let λ be an eigenvalue of A and let $\mathbf{x}$ be a corresponding eigenvector. Then, for each $i \in \{1,\ldots,n\}$, we have

$$\sum_{j=1}^{n} a_{i,j}x_j = \lambda x_i$$

Since $\mathbf{x}$ is non-zero, there is an integer k such that $0 < |x_k| = \max\{|x_i| : i \in \{1,\ldots,n\}\}$. For this k, we have

$$(\lambda - a_{k,k})x_k = \sum_{j=1,j\neq k} a_{k,j}x_j$$

Taking absolute values and applying the triangle inequality:

$$|\lambda - a_{k,k}|\,|x_k| \leq \sum_{j=1,j\neq k} |a_{k,j}|\,|x_j| \leq \sum_{j=1,j\neq k} |a_{k,j}|\,|x_k|\,.$$

Therefore,

$$|\lambda - a_{k,k}| \leq \sum_{j=1,j\neq k}^{n} |a_{k,j}|\,.$$

□

Corollary 1.2. *Let $A \in \mathbb{C}^{n\times n}$. Then,*

$$\rho(A) \leq \max\left\{\sum_{j=1}^{n} |a_{i,j}| : i \in \{1,\ldots,n\}\right\}.$$

1.1 Properties of M-Matrices

In this section, we will introduce several properties of M-matrices. Since all Laplacian matrices are symmetric M-matrices, that will be our focus.

Proposition 1.3. *Let A be a symmetric M-matrix. Then, the eigenvalues of A are non-negative.*

Proof. Since A is an M-matrix, it can be written in the form

$$A = sI - B,$$

where $s \geq \rho(B)$. Let λ be an eigenvalue of A and let $\mathbf{x}$ be a corresponding eigenvector. Then, $A\mathbf{x} = \lambda\mathbf{x}$, which can be re-written as

$$B\mathbf{x} = (s - \lambda)\mathbf{x}.$$

Since λ is real, it follows that $s - \lambda$ is real. Furthermore, $s - \lambda \leq \rho(B)$, which implies that $\lambda \geq 0$. □

The next property (see Corollary 1.5) will be proved using the Perron Frobenius theorem, which we state (in part) below. Note that a matrix A is *irreducible* if there is no permutation matrix P such that

$$PAP^T = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}.$$

For symmetric matrices, A is irreducible if there is no permutation matrix P such that

$$PAP^T = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix},$$

i.e., the graph corresponding to A is connected (does not have more than one connected component). Also, note that an eigenvalue λ of the matrix A is *simple* if there is only one linearly independent eigenvector of A corresponding to λ.

Theorem 1.4 (Perron Frobenius)**.** *If $A \geq 0$ is irreducible, then $\rho(A)$ is a simple eigenvalue.*

Corollary 1.5. *Let A be a singular, irreducible M-matrix. Then, 0 is a simple eigenvalue of A.*

Proof. Since A is an M-matrix, it follows that

$$A = sI - B,$$

where $s \geq \rho(B)$ and $B \geq 0$. Furthermore, since A is irreducible, it follows that $B \geq 0$ is irreducible. Hence, the Perron Frobenius Theorem implies that $\rho(B)$ is a simple eigenvalue.

Note that the eigenvalues of A are of the form $s - \lambda$, where λ is an eigenvalue of B. Since A is singular, it follows that $0 = s - \rho(B)$ is an eigenvalue of A. Hence, $s = \rho(B)$. Furthermore, since $\rho(B)$ is a simple eigenvalue of B, it follows that 0 is a simple eigenvalue of A. □

2 Exercises

I. Let $G = (V, E)$ be a graph and let L be the Laplacian matrix of G. Show that the multiplicity of 0 as an eigenvalue of G is equal to the number of connected components of G.