Graph Theory

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1 Key Topics

Today, we begin our investigation of spectral graph theory. In particular, we show that certain graphs are characterized by their Laplacian spectrum.

Let $G = (V, E)$ be a simple graph and let L be the Laplacian matrix of G. One aspect of spectral graph theory is to determine what properties of the graph G can be realized by the spectrum (eigenvalues) of L. As an example, the multiplcity of the zero eigenvalue of L is equal to the number of connected components of G. Throughout, we will use $\sigma(L)$ to denote the spectrum of the Laplacian matrix L.

1.1 Graphs Determined by Laplacian Spectrum

In a few cases, the graph is completely characterized by its Laplacian spectrum. For example,the empty graph is completely characterized by its Laplacian spectrum.

Proposition 1.1. *Let $G = (V, E)$ be a simple graph and let L be the Laplacian matrix of G. Then, G is the empty graph if and only if $\sigma(L) = \{0, \ldots, 0\}$.*

Proof. If G is the empty graph, the L is the zero matrix and $\sigma(L) = \{0, \ldots, 0\}$.

If $\sigma(L) = \{0, \ldots, 0\}$, then G has n connected components so G must the be the empty graph. □

Next, we show that the complete graph is completely characterized by its Laplacian spectrum. To this end, we need the following property of eigenvalues.

Proposition 1.2. *Let $A \in \mathbb{C}^{n\times n}$ have spectrum $\sigma(A) = \{\lambda_1, \ldots, \lambda_n\}$. Then,*

(i.) $\det(A) = \lambda_1 \cdots \lambda_n$,

(ii.) $\operatorname{tr}(A) = \lambda_1 + \cdots + \lambda_n$.

Theorem 1.3. *Let $G = (V, E)$ be a simple graph and let L be the Laplacian. Then, G is the complete graph of order n if and only if $\sigma(L) = \{0, n, \ldots, n\}$.*

Proof. If G is the complete graph of order n, then L can be written as

$$L = nI - J,$$

where J is the all ones matrix. Let $\mathbf{v}_1$ denote the all ones vector and let $\mathbf{v}_2, \ldots, \mathbf{v}_n$ denote real vectors orthogonal to $\mathbf{v}_1$. Then,

$$L\mathbf{v}_1 = n\mathbf{v}_1 - n\mathbf{v}_1 = 0\mathbf{v}_1$$

and

$$L\mathbf{v}_i = n\mathbf{v}_i,$$

for $i = 2, \ldots, n$.

Conversely, suppose that $\sigma(L) = \{0, n, \ldots, n\}$. Then, G is connected, since the zero eigenvalue has multiplicity one. Moreover,

$$\operatorname{tr}(L) = n(n-1) = 2|E|,$$

which implies that $|E| = \binom{n}{2}$. Hence, G is the complete graph of order n. □

Theorem 1.4. *Let $G = (V, E)$ be a simple graph and let L be the Laplacian. Then, G is the star graph of order n if and only if $\sigma(L) = \{0, 1, \ldots, 1, n\}$.*

Proof. Suppose that G is the star graph of order n. Then, the Laplacian matrix can be written as

$$L = nI - \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & (n-1) & & \\ \vdots & & \ddots & \\ 1 & & & (n-1) \end{bmatrix}.$$

Let $\mathbf{v}_1$ denote the all ones vector. Then, $L\mathbf{v}_1 = n\mathbf{e} - n\mathbf{e} = 0\mathbf{v}_1$. Next, let

$$\mathbf{v}_n = \begin{bmatrix} (n-1) \\ -1 \\ \vdots \\ -1 \end{bmatrix}.$$

Then, $L\mathbf{v}_2 = n\mathbf{v}_n - 0\mathbf{v}_n = n\mathbf{v}_n$. Finally, for $i = 2, \ldots, n-1$, let

$$\mathbf{v}_i = \mathbf{e}_i - \mathbf{e}_{i+1}.$$

Then, $L\mathbf{v}_i = n\mathbf{v}_i - (n-1)\mathbf{v}_i = 1\mathbf{v}_i$.

Conversely, suppose that $\sigma(L) = \{0, 1, \ldots, 1, n\}$. Use the fact that $L + \overline{L} = nI - J$ to prove that G is the star graph of order n. □

2 Exercises

I. Let $G = (V, E)$ be a simple graph and let L be the Laplacian. Also, let $\overline{G}$ denote the complement of G and let $\overline{L}$ denote the Laplacian. Show that $L + \overline{L} = nI - J$.