Graph Theory

Thomas R. Cameron

March 18, 2024

1 Key Topics

Today, we begin our investigation of spectral graph theory. In particular, we complete our proof of the fact that the star graph is characterized by its Laplacian spectrum. Moreover, we use the Laplacian spectrum to provide bounds on the minimum and maximum degree of the graph.

Let $G = (V, E)$ be a simple graph and let L be the Laplacian matrix of G. Also, let $\overline{G}$ denote the complement of G and let $\overline{L}$ denote its Laplacian matrix. Note that $L + \overline{L} = nI - J$, where J denotes the all ones matrix.

Theorem 1.1. *Let $G = (V, E)$ be a simple graph and let L be the Laplacian. Then, G is the star graph of order n if and only if $\sigma(L) = \{0, 1, \ldots, 1, n\}$.*

Proof. Suppose that G is the star graph of order n. Then, the Laplacian matrix can be written as

$$L = nI - \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & (n-1) & & \\ \vdots & & \ddots & \\ 1 & & & (n-1) \end{bmatrix}.$$

Let $\mathbf{v}_1$ denote the all ones vector. Then, $L\mathbf{v}_1 = n\mathbf{e} - n\mathbf{e} = 0\mathbf{v}_1$. Next, let

$$\mathbf{v}_n = \begin{bmatrix} (n-1) \\ -1 \\ \vdots \\ -1 \end{bmatrix}.$$

Then, $L\mathbf{v}_2 = n\mathbf{v}_n - 0\mathbf{v}_n = n\mathbf{v}_n$. Finally, for $i = 2, \ldots, n-1$, let

$$\mathbf{v}_i = \mathbf{e}_i - \mathbf{e}_{i+1}.$$

Then, $L\mathbf{v}_i = n\mathbf{v}_i - (n-1)\mathbf{v}_i = 1\mathbf{v}_i$.

Conversely, suppose that $\sigma(L) = \{0, 1, \ldots, 1, n\}$. Use the fact that $L + \overline{L} = nI - J$ to prove that G is the star graph of order n. $\square$

1.1 Degree Bounds from Laplacian Spectrum

Let $G = (V, E)$ be a simple graph and let L denote its Laplacian spectrum. Denote the spectrum of L by

$$0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n.$$

Also, let δ denote the minimum degree of G and let G denote the maximum degree of G. In 1973, Miroslav Fiedler proved that

$$\lambda_2 \leq \frac{n}{n-1}\delta \text{ and } \lambda_n \geq \frac{n}{n-1}\Delta. \tag{1}$$

Today, we will prove the first inequality in (1). To this end, we will make use of the Courant-Fischer-Weyl min-max principle, which implies that

$$\lambda_2 = \min\left\{\mathbf{x}^T L\mathbf{x} \colon \mathbf{x}^T\mathbf{x} = 1,\ \mathbf{x}^T\mathbf{e} = 0\right\}.$$

Proof. Define

$$\hat{L} = L - \lambda_2 \left(I - n^{-1} J \right)$$

and let $\mathbf{y} = c_1 \mathbf{e} + c_2 \mathbf{x}$, where $\mathbf{x}^T \mathbf{x} = 1$ and $\mathbf{x}^T \mathbf{e} = 0$. Then,

$$\begin{aligned} \mathbf{y}^T \hat{L} \mathbf{y} &= \mathbf{y}^T L \mathbf{y} - \lambda_2 \left(\mathbf{y}^T \mathbf{y} - n^{-1} \mathbf{y}^T J \mathbf{y} \right) \\ &= c_2^2 \mathbf{x}^T L \mathbf{x} - \lambda_2 \left(c_2^2 \mathbf{x}^T \mathbf{x} \right) \\ &= c_2^2 \left(\mathbf{x}^T L \mathbf{x} - \lambda_2 \right) \geq 0. \end{aligned}$$

Since $\mathbf{e}_i = \frac{1}{n} \mathbf{e} + \left(\mathbf{e}_i - \frac{1}{n} \mathbf{e} \right)$, it follows that the diagonal entries of $\hat{L}$ are all non-negative. Therefore,

$$\delta - \lambda_2 \left(1 - \frac{1}{n} \right) \geq 0$$

and the result follows. □

2 Exercises

Let $G = (V, E)$ be a simple graph and let $\overline{G}$ denote the complement of G. Also, let L denote the Laplacian of G and let $\overline{L}$ denote the Laplacian of $\overline{G}$. Recall that $L + \overline{L} = nI - J$, where J denotes the all ones matrix.

I. Write the spectrum of L as

$$0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n.$$

Let $\mathbf{v}_1 = \mathbf{e}, \mathbf{v}_2, \ldots, \mathbf{v}_n$ denote corresponding orthogonal eigenvectors of L. Prove that, for $i = 2, \ldots, n$, $\mathbf{v}_i$ is an eigenvector of $\overline{L}$ corresponding to the eigenvalue $(n - \lambda_i)$. Note that $\overline{L}\mathbf{e} = 0$.

II. Use the previous result to show that the first inequality in (1) implies the second inequality.