Graph Theory

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March 18, 2024

1 Key Topics

Today, we continue our investigation of the algebraic connectivity of a graph.

Let $G = (V, E)$ be a simple graph and let L be the Laplacian matrix of G. Denote the Laplacian spectrum by

$$0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n,$$

where λ_1 is the zero eigenvalue corresponding to the all ones eigenvector.

The algebraic connectivity of G, denote $a(G)$, is defined as the second smallest Laplacian eigenvalue, that is, $a(G) = \lambda_2$. Moreover, using the Courant-Fischer-Weyl min-max principle, we can define the algebraic connectivity as

$$a(G) = \min\{\mathbf{x}^T L\mathbf{x} \colon \mathbf{x}^T\mathbf{x} = 1,\ \mathbf{x}^T\mathbf{e} = 0\}.$$

Last time, we proved the following results.

Theorem 1.1. *Let $G = (V, E)$ be a simple graph and let $\overline{G}$ denote the complement of G. Also, let L and $\overline{L}$ denote the Laplacian of G and $\overline{G}$, respectively. Finally, let the spectrum of L be denoted by*

$$0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n.$$

Then, the spectrum of $\overline{L}$ is

$$0 = \lambda_1 \leq (n - \lambda_n) \leq (n - \lambda_{n-1}) \leq \cdots \leq (n - \lambda_2).$$

Proposition 1.2. *Let $G = (V, E)$ be a simple graph and let $\delta(G)$ denote the minimum degree of G. Then, the algebraic connectivity satisfies*

$$a(G) \leq \frac{n}{n-1}\delta(G).$$

1.1 More Bounds on the Algebraic Connectivity

We begin by defining a related graph parameter: $b(G) = \lambda_n$. By Theorem 1.1, $b(G) = n - a(\overline{G})$. Moreover, using the Courant-Fisher-Weyl min-max principle, we have

$$b(G) = \max\{\mathbf{x}^T L\mathbf{x} \colon \mathbf{x}^T\mathbf{x} = 1,\ \mathbf{x}^T\mathbf{e} = 0\}.$$

Using Theorem 1.1 and Proposition 1.2, we can prove the following inequality

$$b(G) \geq \frac{n}{n-1}\Delta(G), \tag{1}$$

where $\Delta(G)$ is the maximum degree of G.

In the following lemma, we use the min-max principle to obtain bounds on the algebraic connectivity for subgraphs.

Lemma 1.3. *Let $G = (V, E)$ be a simple graph and let $G' = (V', E')$ be a subgraph of G. If $V = V'$, then*

$$a(G') \leq a(G) \text{ and } b(G') \leq b(G).$$

Proof. Let $\hat{G} = (\hat{V}, \hat{E})$, where $V = \hat{V}$ and $\hat{E} = E \setminus E'$. Then, $G = G' \cup \hat{G}$ is an edge-disjoint union. Therefore,

$$L(G) = L(G') + L(\hat{G}),$$

which implies that

$$a(G) \geq a(G') + a(\hat{G}) \geq a(G').$$

Now, let $\overline{G}$ and $\overline{G'}$ denote the complement of G and G', respectively. Since G' is a subgraph of G, where $V = V'$, it follows that $\overline{G}$ is a subgraph of $\overline{G'}$, where $V = V'$. Therefore,

$$a(\overline{G}) \leq a(\overline{G'}),$$

which implies that

$$b(G) \geq b(G').$$

□

Finally, we obtain another bound on the algebraic connectivity and the independence number

Theorem 1.4. *Let $G = (V, E)$ be a simple graph with independence number $\alpha(G) \geq 2$. Then, the algebraic connectivity satisfies*

$$a(G) \leq n - \alpha(G).$$

Proof. Since G has an independent set of $\alpha(G)$ vertices, it follows that $\overline{G}$ has $K_{\alpha(G)}$ as a subgraph. Since $b(K_{\alpha(G)}) = \alpha(G)$, the result follows from Lemma 1.3. □

2 Exercises

I. Prove inequality 1.

II. Identify a family of graphs for which the bound in Theorem 1.4 is sharp.