Graph Theory

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1 Key Topics

Today, we introduce the maximum nullity of a graph. For further information and references, see [3].

Let $G = (V, E)$ be a simple graph. Recall that the set of matrices associated with G is defined by

$$\mathcal{S}(G) = \left\{A \in \mathbb{R}^{n\times n} \colon A^T = A,\ \forall i \neq j a_{ij} \neq 0 \Leftrightarrow \{i,j\} \in E\right\}.$$

Also, recall that the nullity of a matrix A, denoted $\text{nullity}\,(A)$, is the dimension of its null space, i.e., it is the maximum number of linearly independent null vectors of A. If A is symmetric, its nullity is equal to the multiplicity of its zero eigenvalue. Finally, recall that the rank of a matrix A, denoted $\text{rank}\,(A)$, is the dimension of its column space, i.e., it is the maximum number of linearly independent column vectors of A. If A is symmetric, its rank is equal to the combined multiplicity of all non-zero eigenvalues. In fact, for all matrices A, $\text{rank}\,(A) + \text{nullity}\,(A) = n$.

Given $G = (V, E)$, we define its *maximum nullity* by

$$M(G) = \max\left\{\text{nullity}\,(A) : A \in \mathcal{S}(G)\right\}.$$

Furthermore, we define its *minimum rank* by

$$mr(G) = \min\left\{\text{rank}\,(A) : A \in \mathcal{S}(G)\right\}.$$

1.1 Families of Graphs

There is at least one graph where the maximum nullity is easy to find, i.e., the empty graph. Note that every matrix $A \in \mathcal{S}(E_n)$ is of the form

$$A = \begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \dots & d_n \end{bmatrix},$$

where $d_1, \ldots, d_n$ are arbitrary real numbers. Hence, if we choose $d_1 = \cdots = d_n = 0$, then A is the zero matrix, which has nullity n. Therefore, $M(E_n) = n$ and $mr(E_n) = 0$.

In fact, the empty graph is the only graph with with maximum nullity n. Indeed, suppose that $G = (V, E)$ is a simple graph with $M(G) = n$. Then, there is an $A \in \mathcal{S}(G)$ with a zero eigenvalue of multiplicity n. By the spectral decomposition of symmetric matrices,

$$A = 0\mathbf{v}_1\mathbf{v}_1^T + \cdots + 0\mathbf{v}_n\mathbf{v}_n^T,$$

where $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are orthonormal eigenvectors of A corresponding to the zero eigenvalue. Hence, A is the all zeros matrix, which implies that G must be the empty graph. We summarize this result in the following proposition.

Proposition 1.1. *Let $G = (V, E)$ be a simple graph of order $n \geq 1$. Then, $G \sim E_n$ if and only if $M(G) = n$.*

1.2 Bounds on the Maximum Nullity

Let $G = (V, E)$ be a simple graph and let L be the Laplacian matrix of G. Also, let k denote the number of connected components of G, then nullity $(k) = k$ Hence,

$$M(G) \geq k \text{ and } mr(G) \leq n - k.$$

Another useful lower bound on $M(G)$ (upper bound on $mr(G)$ comes from the clique cover number of a graph. Recall that a clique of G is any subset of vertices $V' \subseteq V$ such that the induced subgraph $G' = (V', E')$, where

$$E' = \{\{u, v\} \colon u, v \in V',\ \{u, v\} \in E\},$$

is a complete graph. Let $V_1, \ldots, V_k$ denote cliques of G and let $G_1, \ldots, G_k$ denote the corresponding subgraphs of G that are complete graphs. Then, $G_1, \ldots, G_k$ denote a *clique cover* of G if every edge of G is contained in at least one G_i. The *clique cover number* of G, denoted $cc(G)$, is the smallest number of cliques in a clique cover of G. The following bound was proven in [2]:

$$M(G) \geq n - cc(G) \text{ and } mr(G) \leq cc(G).$$

Finally, we note that for trees, the maximum nullity is equal to the path cover number, this result was proven in [1]. A *path cover* of G is a collection of induced paths where every vertex in G is in exactly one path. The *path cover number* of G, denoted $P(G)$, is the minimum number of induced paths to form a path cover of G.

2 Exercises

I. Prove that $M(K_n) = n - 1$. Is this the only family of graphs of order $n \geq 1$ with maximum nullity $n - 1$.

II. Use the clique cover number to determine lower and upper bounds on the maximum nullity and minimum rank, respectively, for the graph (left) in Figure 1.

III. Use the path cover number to determine the maximum nullity (and minimum rank) of the tree (right) in Figure 1.

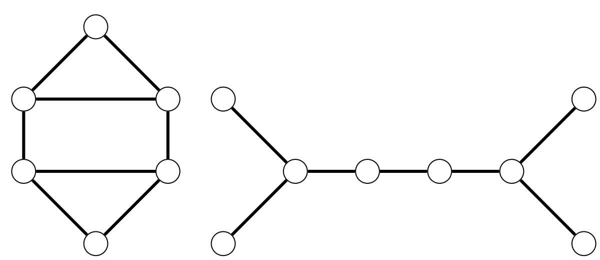

Figure 1: Graphs for exercise 2 and 3.

References

[1] F. Barioli, S. Fallat, and L. Hogben, *Computation of minimal rank and path cover number for graphs*, Linear Algebra Appl., 392 (2004), pp. 289–303.

[2] S. Fallat and L. Hogben, *The minimum rank of symmetric matrices described by a graph: A survey*, Linear Algebra Appl., 426 (2007), pp. 558–582.

[3] L. Hogben, J. C.-H. Lin, and B. Shader, *Inverse Problems and Zero Forcing for Graphs*, AMS, Providence, Rhode Island, 2022.