Graph Theory

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1 Key Topics

Today, we continue our investigation of the maximum nullity of a graph and introduce the zero-forcing coloring game, which will lead to an upper bound on the maximum nullity. For further information and references, see [3].

Let $G = (V, E)$ be a simple graph. Recall that the set of matrices associated with G is defined by

$$\mathcal{S}(G) = \left\{A \in \mathbb{R}^{n\times n} \colon A^T = A,\ \forall i \neq j a_{ij} \neq 0 \Leftrightarrow \{i,j\} \in E\right\}.$$

Also, the maximum nullity of G is defined by

$$M(G) = \max\left\{\text{nullity}\,(A) : A \in \mathcal{S}(G)\right\}$$

and the minimum rank of G is defined by

$$mr(G) = \min\left\{\text{rank}\,(A) : A \in \mathcal{S}(G)\right\}.$$

1.1 Zero-Forcing

Zero forcing is a coloring game on a graph, where vertices are either colored or non-colored. An initial set of colored vertices can force non-colored vertices to become colored following a color change rule. While there are many color change rules, see [3, Chapter 9], we will use the *standard rule* which states that a colored vertex u can *force* a non-colored vertex v if v is the only non-colored neighbor of u.

A zero forcing game on G corresponds to a collection of subsets $C^{(i)}$ and $C^{[i]}$, $i \geq 0$, where $C = C^{(0)} = C^{[0]}$ is the set of initially colored vertices, $C^{(i)}$ denotes the set of vertices that are forced at time step i, and $C^{[i]}$ denotes the set of all colored vertices after time step i. Note that

$$C^{[i+1]} = C^{[i]} \cup C^{(i+1)},\ i \geq 0.$$

Furthermore, every vertex of G is in exactly one set $C^{(i)}$ and if $v \in C^{(i+1)}$ then v must be forced by exactly one of its neighbors u such that u and all of its neighbors except for v are in $C^{[i]}$.

Since the graph is finite, there exists a $t \geq 0$ for which $C^{[t]} = C^{[t+i]}$, for all $i \geq 0$; we reference $C^{[t]}$ as the *final coloring* of C. The final coloring of C is also refereed to as the *closure* of C, denoted $\text{cl}\,(C)$, for example see [2]. If $\text{cl}\,(C) = V$, then we say that C is a *zero forcing set* of G. The *zero forcing number* of G is defined as

$$\text{Z}(G) = \min\left\{|C| : \text{cl}\,(C) = V\right\}.$$

If C is a zero forcing set of G such that $\text{Z}(G) = |C|$, we say that C is a minimum zero forcing set.

As an example, consider the zero-forcing game shown in Figure 1. Note that

$$C = \{2,4,5\},\ C^{[1]} = \{2,3,4,5\},\ C^{[2]} = \{1,2,3,4,5,6\}.$$

Since $C^{[2]} = V$, we know that C is a zero-forcing set of G. Furthermore, since no set of 2 colored vertices can force the entire graph, it follows that $\text{Z}(G) = 3$. Also, note that $C = \{2,4,5\}$ is not the only minimum zero forcing set of G; for example, $C = \{1,4,5\}$ and $C = \{4,5,6\}$ are also minimum zero forcing sets. What are all the minimum zero forcing sets of G?

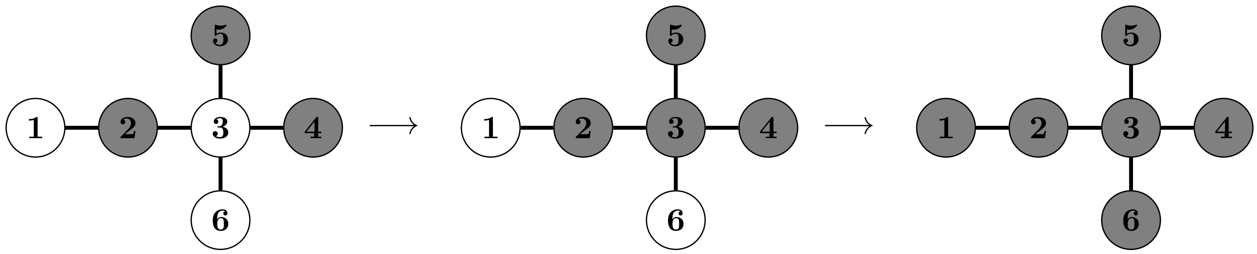

Figure 1: Zero-forcing game on a graph

For $i \geq 0$, the vertices in $C^{(i+1)}$ were each forced by some vertex in $C^{[i]}$. If the vertex u forced the vertex v, we write $u \to v$. Given an initial coloring C, we let $\mathcal{F}(C)$ denote a *set of forces* that produce the final coloring. Note that there may be more than one set of forces that produce the final coloring. Given a set $\mathcal{F}(C)$ of forces, a *forcing chain* is a maximal sequence of vertices $(v_1, v_2, \ldots, v_s)$ such that for $i = 1, \ldots, s-1$, $v_i \to v_{i+1}$. If $v_1 \in C$ does not force, then (v_1) is a forcing chain. For example, the zero-forcing game in Figure 1 may correspond to the set of forces

$$\mathcal{F} = \{5 \to 3, 2 \to 1, 3 \to 6\}$$

and the following forcing chains

$$(5, 3, 6),\ (2, 1),\ (4).$$

What other set of forces could correspond to the zero-forcing game in Figure 1 and what are the implied forcing chains? No matter which set of forces we choose, the forcing chains induce a path cover of the graph G. Hence, $\mathrm{Z}(G) \geq P(G)$.

1.2 Families of Graphs

In this section, we state (and prove) several well-known results on the zero-forcing number for certain families of graphs. Most of these results were established in [1].

Theorem 1.1. *a.* $\mathrm{Z}(E_n) = n$

b. $\mathrm{Z}(K_n) = n - 1$

c. $\mathrm{Z}(P_n) = 1$

d. $\mathrm{Z}(C_n) = 2$

e. For all trees T, $\mathrm{Z}(T) = P(T)$.

2 Exercises

I. Prove Theorem 1.1.

II. Which, if any, zero-forcing values characterize the graph.

References

[1] AIM Minimum Rank – Special Graphs Work Group, *Zero forcing sets and the minimum rank of graphs*, Linear Algebra and its Applications, 428 (2008), pp. 1628–1648.

[2] B.Brimkov, C. C. Fast, and I. V. Hicks, *Computational approaches for zero forcing and related problems*, European Journal of Operational Research, 273 (2019), pp. 889–903.

[3] L. Hogben, J. C.-H. Lin, and B. Shader, *Inverse Problems and Zero Forcing for Graphs*, AMS, Providence, Rhode Island, 2022.