Graph Theory

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1 Key Topics

Today, we continue our investigation of the zero forcing number of a graph. For further information and references, see [2].

Let $G = (V, E)$ be a simple graph. Recall that zero forcing is a coloring game on G where an initial set of colored vertices can force non-colored vertices to become colored. In particular, a colored vertex u can force a non-colored vertex v if v is the only non-colored neighbor of u.

A zero forcing game on G corresponds to a collection of subsets $C^{(i)}$ and $C^{[i]}$, $i \geq 0$, and a collection of forces $\mathcal{F}$. In particular, $C = C^{(0)} = C^{[0]}$ is the set of initially colored vertices, $C^{(i)}$ denotes the set of vertices that are forced at time step i, $C^{[i]}$ denotes the set of all colored vertices after time step i, and $\mathcal{F}$ denotes all forces. Note that

$$C^{[i+1]} = C^{[i]} \cup C^{(i+1)},\ i \geq 0.$$

Furthermore, every vertex of G is in exactly one set $C^{(i)}$ and if $v \in C^{(i+1)}$ then v must be forced by exactly one of its neighbors u such that u and all of its neighbors except for v are in $C^{[i]}$; in this case, the forcing $u \to v$ is in $\mathcal{F}$.

Since the graph is finite, there exists a $t \geq 0$ for which $C^{[t]} = C^{[t+i]}$, for all $i \geq 0$; we reference $C^{[t]}$ as the final coloring of C. The final coloring of C is also refereed to as the closure of C, denoted $\operatorname{cl}(C)$. If $\operatorname{cl}(C) = V$, then we say that C is a zero forcing set of G. The zero forcing number of G is defined as

$$\mathrm{Z}(G) = \min\left\{|C| :\ \operatorname{cl}(C) = V\right\}.$$

If C is a zero forcing set of G such that $\mathrm{Z}(G) = |C|$, we say that C is a minimum zero forcing set.

1.1 Zero Forcing on Trees

Given a set of forces $\mathcal{F}$, a forcing chain is a maximal sequence of vertices $(v_1, \ldots, v_s)$ such that for $i = 1, \ldots, s-1$, $v_i \to v_{i+1}$. If $v_1 \in C$ does not force, then (v_1) is a forcing chain. Note that each chain induces a path on G. Furthermore, if C is a zero forcing set, then every vertex is in exactly on chain. Hence, if C is a zero forcing set, then a collection of forcing chains will induce a path cover of G of size $|C|$. Therefore, $\mathrm{Z}(G) \geq P(G)$ is the path cover number of G. In what follows, we prove that this bound is sharp for trees.

Theorem 1.1 (Proposition 4.2 in [1]). *Let T be a tree of order $n \geq 1$. Then, $\mathrm{Z}(T) = P(T)$.*

Proof. We proceed via induction on $P(T)$ to show that a zero forcing set of T can be constructed by choosing a minimum path cover and selecting either the initial or terminal vertex (not both) of each path. If $P(T) = 1$, then this construction is obvious since T is a path graph of order n. Let $k \geq 1$ and assume that such a zero forcing set can be constructed for all trees with a path cover number of k. Let T denote a tree with a path cover number of $k+1$ and choose a minimum path cover of T. Let C denote a set of initially colored vertices consisting of one initial or terminal vertex (not both) of each path. Also, identify a path (denoted P') in the minimum path cover that is connected to the rest of T (denoted T') by a single edge $\{u, v\}$, where $u \in P'$ and $v \in T'$.

Then, T' is a tree with $P(T') = k$ and $C' = C \cap V(T')$ is a zero forcing set of T' by the induction hypothesis. Since P' has its initial or terminial vertex colored, all vertices up to v can be forced in P'. At which point, the vertices in C' can force all of T'. Finally, the vertex v can force the rest of the vertices in P'. Therefore, C is a zero forcing set of T. $\square$

1.2 Zero Forcing and Maximum Nullity

Zero forcing was shown to be an upper bound on the maximum nullity of a graph [1]. The motivation behind this result is the following observation.

Observation 1.2. Let $G = (V, E)$ be a simple graph. If

- $C \subseteq V$,
- $A \in \mathcal{S}(G)$,
- $A\mathbf{x} = 0$,
- $x_i = 0$ for all $i \in C$,
- $i \in C$, $j \notin C$, and j is the only vertex in $N(i)$ that is not in C,

then $x_j = 0$.

One can use this observation to apply the game of zero forcing on G to the process of forcing zeros in a null vector of $A \in \mathcal{S}(G)$. As an example, consider the zero-forcing game shown in Figure 1. Note that

$$C = \{2, 4, 5\},\ C^{[1]} = \{2, 3, 4, 5\},\ C^{[2]} = \{1, 2, 3, 4, 5, 6\}.$$

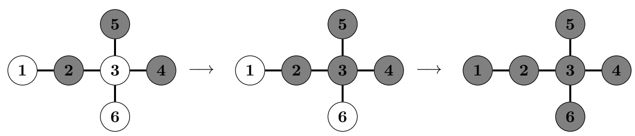

Figure 1: Zero-forcing game on a graph

Let $A \in \mathcal{S}(G)$, where G is the graph in Figure 1, and let $\mathbf{x}$ be a null vector of A, where $x_i = 0$ for all $i \in C$. Then,

$$A = \begin{bmatrix} d_1 & a_{12} & 0 & 0 & 0 & 0 \\ a_{21} & d_2 & a_{23} & 0 & 0 & 0 \\ 0 & a_{32} & d_3 & a_{34} & a_{35} & a_{36} \\ 0 & 0 & a_{43} & d_4 & 0 & 0 \\ 0 & 0 & a_{53} & 0 & d_5 & 0 \\ 0 & 0 & a_{63} & 0 & 0 & d_6. \end{bmatrix} \text{ and } \mathbf{x} = \begin{bmatrix} x_1 \\ 0 \\ x_3 \\ 0 \\ 0 \\ x_6 \end{bmatrix}.$$

Furthermore, row 5 of $A\mathbf{x} = 0$ implies that $a_{35}x_3 = 0$, which further implies that $x_3 = 0$. Let

$$\mathbf{x}^{[1]} = [x_1,\ 0,\ 0,\ 0,\ 0,\ x_6]^T.$$

Then, row 2 of $A\mathbf{x}^{[1]} = 0$ implies that $a_{21}x_1 = 0$ and row 3 of $A\mathbf{x}^{[1]}$ implies that $a_{36}x_6 = 0$. Hence, $x_1 = 0$ and $x_6 = 0$, which implies that $\mathbf{x} = 0$.

In general, we have the following result.

Proposition 1.3. *Let $G = (V, E)$ be a simple graph and let $C \subseteq V$ denote a zero-forcing set of G. Also, let $A \in \mathcal{S}(G)$ and let $\mathbf{x}$ denote a null vector of A such that $x_i = 0$ for all $i \in C$. Then, $\mathbf{x} = 0$.*

2 Exercises

I. Prove Proposition 1.3.

References

[1] AIM Minimum Rank – Special Graphs Work Group, *Zero forcing sets and the minimum rank of graphs*, Linear Algebra and its Applications, 428 (2008), pp. 1628–1648.

[2] L. Hogben, J. C.-H. Lin, and B. Shader, *Inverse Problems and Zero Forcing for Graphs*, AMS, Providence, Rhode Island, 2022.