Graph Theory

Thomas R. Cameron

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1 Key Topics

Today, we continue our proof that the zero forcing number of a graph is an upper bound on its maximum nullity. For further information and references, see [2].

Let $G = (V, E)$ be a simple graph. Recall that zero forcing is a coloring game on G where an initial set of colored vertices can force non-colored vertices to become colored. In particular, a colored vertex u can force a non-colored vertex v if v is the only non-colored neighbor of u.

Last time we applied the game of zero forcing on G to the process of forcing zeros in a null vector of $A \in \mathcal{S}(G)$. As an example, consider the zero forcing game shown in Figure 1. Note that

$$C = \{1, 5, 6\},\ C^{[1]} = \{1, 4, 5, 6\},\ C^{[2]} = \{1, 2, 4, 5, 6\},\ C^{[3]} = \{1, 2, 3, 4, 5, 6\}.$$

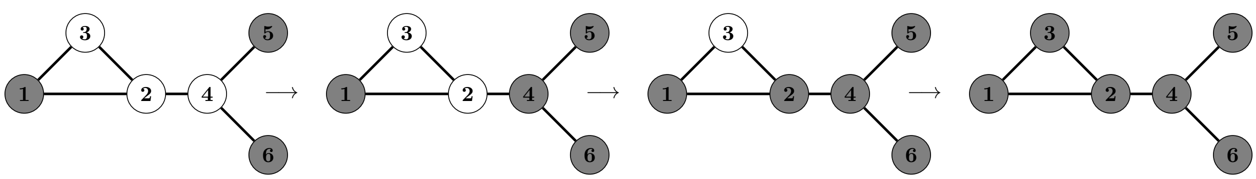

Figure 1: Zero-forcing game on a graph

Let $A \in \mathcal{S}(G)$, where G is the graph in Figure 1, and let $\mathbf{x}$ be a null vector of A, where $x_i = 0$ for all $i \in C$. Then,

$$A = \begin{bmatrix} d_1 & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{21} & d_2 & a_{23} & a_{24} & 0 & 0 \\ a_{31} & a_{32} & d_3 & 0 & 0 & 0 \\ 0 & a_{42} & 0 & d_4 & a_{45} & a_{46} \\ 0 & 0 & 0 & a_{54} & d_5 & 0 \\ 0 & 0 & 0 & a_{64} & 0 & d_6 \end{bmatrix} \text{ and } \mathbf{x} = \begin{bmatrix} 0 \\ x_2 \\ x_3 \\ x_4 \\ 0 \\ 0 \end{bmatrix}.$$

Furthermore, row 5 of $A\mathbf{x} = 0$ implies that $a_{54}x_4 = 0$, which further implies that $x_4 = 0$. Let

$$\mathbf{x}^{[1]} = [0, x_2, x_3, 0, 0, 0]^T.$$

Then, row 4 of $A\mathbf{x}^{[1]} = 0$ implies that $a_{42}x_2 = 0$, i.e., $x_2 = 0$. Let

$$\mathbf{x}^{[2]} = [0, 0, x_3, 0, 0, 0]^T.$$

Then, row 2 of $A\mathbf{x}^{[2]} = 0$ implies that $a_{23}x_3 = 0$, i.e., $x_3 = 0$. Therefore, $\mathbf{x} = 0$.

In general, we have the following result.

Proposition 1.1. *Let $G = (V, E)$ be a simple graph and let $C \subseteq V$ denote a zero-forcing set of G. Also, let $A \in \mathcal{S}(G)$ and let $\mathbf{x}$ denote a null vector of A such that $x_i = 0$ for all $i \in C$. Then, $\mathbf{x} = 0$.*

1.1 Zero Forcing and Maximum Nullity

It turns out that Proposition 1.1 implies a bound on the nullity of a matrix $A \in \mathcal{S}(G)$. To make this connection, we first consider the following observation.

Observation 1.2. If U and V are subspaces of $\mathbb{R}^n$ and $\dim V > n - \dim U$, then U and V intersect non-trivially, i.e., there exists a non-zero $\mathbf{x} \in U \cap V$.

As an illustration of Observation 1.2, note that every 2 dimensional subspace of $\mathbb{R}^3$ is a plane that contains the origin. If U and V are both planes in $\mathbb{R}^3$, then they must intersect at infinitely many points.

Lemma 1.3. *Suppose $\alpha \subseteq \{1, \ldots, n\}$ with $|\alpha| < k$ and U is a subspace of $\mathbb{R}^n$ with $\dim U = k$. Then, U contains a non zero vector $\mathbf{v}$, where $v_i = 0$ for all $i \in \alpha$.*

Proof. Let V denote the subspace of vectors $\mathbf{v}$, where $v_i = 0$ for all $i \in \alpha$. Let $S = \{\mathbf{e}_j : j \notin \alpha\}$. Then, S forms a basis for V and it follows that

$$\dim V = |S| > n - k,$$

since $|\alpha| < k$. By Observation 1.2, U and V intersect non-trivially. Therefore, there exists a non zero vector $\mathbf{v}$ in U such that $v_i = 0$ for all $i \in \alpha$. □

We are now ready to prove that the zero forcing number of a graph is an upper bound on the maximum nullity.

Theorem 1.4 (Proposition 2.4 in [1]). *Let $G = (V, E)$ be a simple graph. Then,* $\mathrm{M}(G) \leq \mathrm{Z}(G)$.

Proof. For the sake of contradiction, suppose that $\mathrm{M}(G) > \mathrm{Z}(G)$. Then, there exists an $A \in \mathcal{S}(G)$ such that $\operatorname{nullity}(A) > \mathrm{Z}(G)$. Let $C \subseteq V$ denote a zero forcing set of G such that

$$|C| = \mathrm{Z}(G) < \operatorname{nullity}(A).$$

Then, Lemma 1.3 implies that there is a non zero vector $\mathbf{v}$ in the null space of A such that $v_i = 0$ for all $i \in C$. However, this contradicts Proposition 1.1. □

References

[1] AIM Minimum Rank – Special Graphs Work Group, *Zero forcing sets and the minimum rank of graphs*, Linear Algebra and its Applications, 428 (2008), pp. 1628–1648.

[2] L. Hogben, J. C.-H. Lin, and B. Shader, *Inverse Problems and Zero Forcing for Graphs*, AMS, Providence, Rhode Island, 2022.