Graph Theory

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1 Key Topics

Today, we discuss computational approaches to the zero forcing number of a graph.

Let $G = (V, E)$ be a simple graph. Recall that zero forcing is a coloring game on G where an initial set of colored vertices can force non-colored vertices to become colored. In particular, a colored vertex u can force a non-colored vertex v if v is the only non-colored neighbor of u. An initial set of colored vertices C is a zero forcing set if repeated application of the color change rule results all of V being colored. The zero forcing number $\mathrm{Z}(G)$ is the minimum cardinality of a zero forcing set.

In [1], the authors prove that the maximum nullity $\mathrm{M}(G)$ is bounded above by the zero forcing number $\mathrm{Z}(G)$. Recall that

$$\mathrm{M}(G) = \max\{\text{nullity}\,(A) : A \in \mathcal{S}(G)\}.$$

Furthermore, in [3], the authors show that $\mathrm{Z}(G) = \mathrm{M}(G)$ for all G of order $n \leq 7$. For this reason, the computation of the zero forcing number of a graph become of great interest to the graph theory and combinatorial optimization community.

1.1 Brute Force

One way to compute the zero forcing number of a graph is to check all possible subsets of vertices, play the game of zero forcing, and record the smallest possible zero forcing set. For a graph of order n, there are 2^n subsets of vertices; hence, this brute force method is not suitable for large graphs. In 2014, an improved brute force search was developed using dynamic programming; this method was dubbed the wavefront algorithm, see [2] for a detailed analysis of this algorithm.

These algorithms are available to you in Sage, see `https://sage.math.iastate.edu/home/pub/84/`.

1.2 Integer Programming

In [2] the first integer programming models were developed for zero forcing. These models identify an objective function and set of constraints that describe the game of zero forcing. Then, a powerful solver such as Gurobi can be used to optimize the objective function over the given set of constraints. For example, the model in (1a)–(1h) is known as the infection model, where T is the maximum number of time steps needed to complete the game of zero forcing.

$$
\begin{aligned}
\text{minimize} \quad & \sum_{v \in V} s_v && \text{(1a)}\\
\text{subject to} \quad & s_v + \sum_{a \in A:\, a=(u,v)} y_a = 1, \quad \forall v \in V, && \text{(1b)}\\
& x_u - x_v + (T+1)y_a \le T, \quad \forall a=(u,v) \in E, && \text{(1c)}\\
& x_w - x_v + (T+1)y_a \le T, \quad \forall a=(u,v) \in E,\ \forall w \in N(u) \setminus \{v\}, && \text{(1d)}\\
& s \in \{0,1\}^n, && \text{(1e)}\\
& x \in \{0,1,\ldots,T\}^n, && \text{(1f)}\\
& y \in \{0,1\}^{2m}, && \text{(1g)}\\
& z \in \{0,1,\ldots,T\} && \text{(1h)}
\end{aligned}
$$

Note that the s_v variables determine which vertices are in the zero forcing set; the x_v variables determine when a vertex is colored; and the y_a variables determine what forcings occur.

As an example, consider the zero forcing game shown in Figure 1. Note that

$$C = \{1,5,6\},\ C^{[1]} = \{1,4,5,6\},\ C^{[2]} = \{1,2,4,5,6\},\ C^{[3]} = \{1,2,3,4,5,6\}.$$

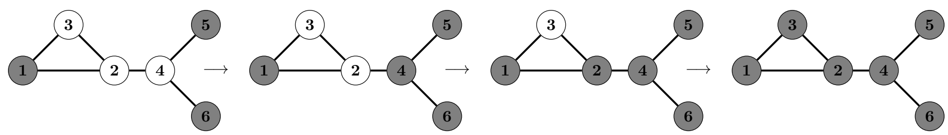

Figure 1: Zero-forcing game on a graph

Moreover, the following variables form a feasible solution for the infection model:

$$s = \{1,0,0,0,1,1\} \tag{2}$$
$$x = \{0,2,3,1,0,0\} \tag{3}$$
$$y = \{0,0,0,0,1,0,0,1,0,1,0,0\} \tag{4}$$

Next week we will discuss the fort cover model for zero forcing, which is shown in (5a)–(5c).

$$
\begin{aligned}
\text{minimize} \quad & \sum_{v \in V} s_v && \text{(5a)}\\
\text{subject to} \quad & \sum_{v \in F} s_v \ge 1, \quad \forall F \in \mathcal{F}_G, && \text{(5b)}\\
& s \in \{0,1\}^n && \text{(5c)}
\end{aligned}
$$

2 Exercises

1. Verify that the variables in (2)–(4) are feasible solutions to the infection model.

References

[1] AIM Minimum Rank – Special Graphs Work Group, *Zero forcing sets and the minimum rank of graphs*, Linear Algebra and its Applications, 428 (2008), pp. 1628–1648.

[2] B.Brimkov, C. C. Fast, and I. V. Hicks, *Computational approaches for zero forcing and related problems*, European Journal of Operational Research, 273 (2019), pp. 889–903.

[3] L. DeLoss, J. Grout, L. Hogben, T. McKay, J. Smith, and T. Tims, *Techniques for determining the minimum rank of a small graph*, Linear Algebra Appl., 432 (2010), pp. 2995–3001.