Graph Theory

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1 Key Topics

Today, we continue our discussion of computational approaches to the zero forcing number of a graph. In particular, we investigate the fort cover model from [1], which is shown in (1a)–(1c).

$$\text{minimize} \quad \sum_{v \in V} s_v \tag{1a}$$

$$\text{subject to} \quad \sum_{v \in F} s_v \geq 1, \quad \forall F \in \mathcal{F}_G, \tag{1b}$$

$$s \in \{0,1\}^n \tag{1c}$$

When compared with the infection model, one is immediately impressed by the simplicity of the fort cover model. However, this simplicity is a bit of a red herring for two reasons: One the fort cover model does not contain all information on the zero forcing game as the infection model, for instance the forcing and time step information is gone. Two, the fort cover model may contain an exponential number of constraints. For this reason, the fort cover model must use constraint generation, see [3, 7] In particular, a relaxed model is obtained from the full model by omitting the constraints in (1b); then, the relaxed model is solved and a set of violated constraints from the full model are added to the relaxed model, and this process is repeated until there are no more violated constraints.

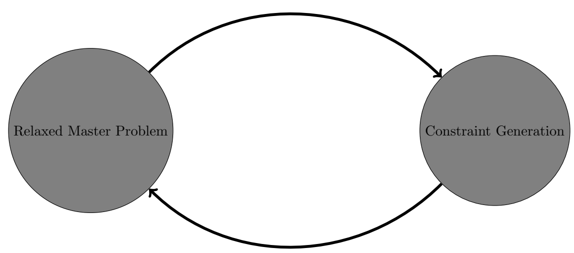

Figure 1: Illustration of constraint generation.

1.1 The Forts of a Graph

Let $G = (V, E)$ be a simple graph. A non-empty subset $F \subseteq V$ is a *fort* if no vertex $u \in V \setminus F$ has exactly one neighbor in F. As an example, we show the forts (white) of C_4 in Figure 2. Note that the vertices not in the fort are colored gray. Since no vertex outside of the fort has exactly one neighbor in the fort, it follows that none of the gray vertices can perform a forcing. Therefore, the set of gray vertices is an example of a stalled failed zero forcing set, where $S \subset V$ is a *stalled failed zero forcicng set* if no vertex in S can force a vertex in $V \setminus S$ [6].

Let $\mathcal{F}_G$ denote the collection of all forts of G. The following theorem shows that $\mathcal{F}_G$ forms a cover for the zero forcing sets of G. While one direction of this result was originally proven in [5, Theorem 3], both directions are shown in [1, Theorem 8].

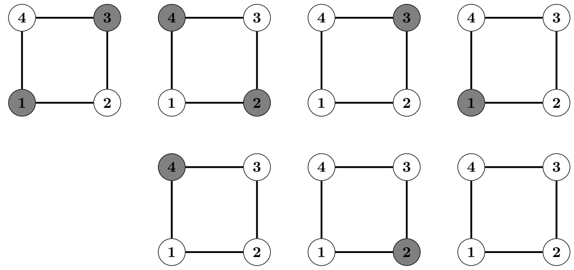

Figure 2: Forts (white) of C_4

Theorem 1.1. *Let $G = (V, E)$ be a simple graph. Then, $S \subseteq V$ is a zero forcing set of G if and only if $S \cap F \neq \emptyset$ for all $F \in \mathcal{F}_G$.*

Proof. Let $S \subseteq V$. Suppose there is an $F \in \mathcal{F}_G$ such that $S \cap F = \emptyset$. Then, color all vertices in S and do not color any vertex in $V \setminus S \supseteq F$. Since no vertex in F can be forced by a vertex outside of F, it follows that S is not a zero forcing set.

Suppose S is not a zero forcing set. Then, $\mathrm{cl}\,(S)$ is a stalled zero forcing set, i.e., $F = V \setminus \mathrm{cl}\,(S)$ is a fort. Therefore, there exists an $F \in \mathcal{F}_G$ such that $S \cap F = \emptyset$. $\square$

Theorem 1.1 motivates the fort cover model; in particular, see constraint (1b). Note that Theorem 1.1 still holds if $\mathcal{F}_G$ is replaced by the minimal forts of G, where $F \in \mathcal{F}_G$ is *minimal* if $F \not\subset A$ for all $A \in \mathcal{F}_G$. For this reason, minimal forts play an important role in the implementation of the fort cover model. Moreover, minimal forts are an important tool for understanding the fractional zero forcing number [2] and the spark of a graph [4].

2 Exercises

1. Identify the minimal forts of the empty, path, cycle, and complete graphs.

References

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