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On the restricted numerical range of the Laplacian matrix for digraphs

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ABSTRACT
In this article, we present the restricted numerical for the Laplacian matrix of a directed graph (digraph). We motivate our interest in the restricted numerical range by its close connection to the algebraic connectivity of a digraph. Moreover, we show that the restricted numerical range can be used to characterize digraphs, some of which are not determined by their Laplacian spectrum. Finally, we identify a new class of digraphs that are characterized by having a real restricted numerical range.

KEYWORDS
numerical range; directed graph; Laplacian; algebraic connectivity

AMS CLASSIFICATION
05C20, 05C50, 15A18, 15A60

1. Introduction

Spectral graph theory has a long and interesting history which intersects the seminal works on the algebraic connectivity, i.e., the second smallest eigenvalue of the graph Laplacian [7,10]. Furthermore, the eigenvalues of the graph Laplacian have been used to characterize graphs and have applications to data mining, image processing, mixing of Markov chains, chromatic numbers, and much more [16–18,20,21,24]. However, there has been far less success in the study of the spectra of directed graphs, which is mainly due to the asymmetry of the associated matrices [15,18].

While there are some results on the spectra of digraphs, see [5] and the references therein, these results are specific to the adjacency matrix or the (symmetric) Laplacian matrix for strongly connected digraphs as defined in [8]. A few notable exceptions are the results in [2,6,22], which apply to the (asymmetric) Laplacian matrix for digraphs.

In this article, we develop a novel approach for characterizing digraphs using the restricted numerical range of the Laplacian matrix. Until now, the numerical range has played only a minor role in graph theory, where it has been used to develop eigenvalue bounds, study the essential spectrum of the Laplace operator, and to model uncertainty curves in spectral graph theory [1,9,19]. We will show that, in fact, the

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restricted numerical range of the Laplacian matrix is teeming with information about the underlying digraph. In particular, in Section 2, we demonstrate that the algebraic connectivity of a digraph can easily be obtained from its restricted numerical range. In Section 3, we classify several digraphs that are characterized by their restricted numerical range but not their Laplacian spectrum. Furthermore, in Section 4, we identify a new class of digraphs that are characterized by having a real restricted numerical range.

2. The Laplacian Matrix and the Restricted Numerical Range

In this section, we define the Laplacian matrix for digraphs and its restricted numerical range. We motivate our interest in the restricted numerical range by illustrating its connection to the algebraic connectivity of a digraph and using it to characterize the empty, complete, and cycle digraphs.

2.1. The Laplacian Matrix for Digraphs

Let $\mathbb{G}$ denote the set of finite simple unweighted digraphs. For each $\Gamma \in \mathbb{G}$, we have $\Gamma = (V, E)$, where $V = \{1, 2, \ldots, n\}$ is the vertex set and $E \subseteq V \times V$ is the edge set, and $(i, j) \in E$ if and only if there is an edge from i to j.

Given $\Gamma \in \mathbb{G}$, we denote the *out-degree* of the vertex $i \in V$ by $d^+(i)$, which is equal to the number of edges of the form $(i, j) \in E$. Similarly, we denote the *in-degree* of the vertex $i \in V$ by $d^-(i)$, which is equal to the number of edges of the form $(j, i) \in E$. We define the *out-degree matrix* as the diagonal matrix D whose ith diagonal entry is equal to $d^+(i)$, for all $i = 1, \ldots, n$. In addition, we define the *adjacency matrix* as the $(0, 1)$ matrix $A = [a_{ij}]_{i,j=1}^n$, where $a_{ij} = 1$ if and only if the edge $(i, j) \in E$. Furthermore, we define the *Laplacian* matrix of Γ by

$$L = D - A.$$

Let $\Gamma = (V, E) \in \mathbb{G}$ and let $V' \subseteq V$. Then, we define the induced subgraph Γ' of Γ as the digraph with vertex set V' and edge set $E' = E \cap (V' \times V')$. A subgraph of Γ is *strongly connected* if for each pair of vertices $i, j \in V'$, either $i = j$ or there is a directed path from i to j and a directed path from j to i. A *strongly connected component* of Γ is a maximal strongly connected subgraph.

A digraph Γ can be uniquely decomposed into strongly connected components. Also, the Laplacian matrix, possibly after re-ordering the vertices, can be written in Frobenius normal form [4]:

$$L = \begin{bmatrix} L_1 & L_{12} & \cdots & L_{1r} \\ & L_2 & \cdots & L_{2r} \\ & & \ddots & \vdots \\ & & & L_r \end{bmatrix}, \tag{1}$$

where the blocks L_k are irreducible matrices that correspond to the strongly connected components Γ_k of Γ. Let V_k denote the vertex set of the strongly connected component Γ_k. Then, the (i, j) entry of the $(0, -1)$ submatrix L_{kl}, $1 \leq k < l \leq r$, is equal to -1 if and only if there is an edge from $i \in V_k$ to $j \in V_l$.

2.2. The Restricted Numerical Range

In general, the numerical range (field of values) of a complex matrix A is defined as follows [13,23]:

$$W(A) = \{\mathbf{x}^* A\mathbf{x} \colon \ \mathbf{x} \in \mathbb{C}^n, \ \ \|\mathbf{x}\| = 1\}.$$

For our purposes, we are interested in the *restricted numerical range* of the Laplacian matrix, which we define by

$$W_r(L) = \{\mathbf{x}^* L\mathbf{x} \colon \ \mathbf{x} \perp \mathbf{e}, \ \ \|\mathbf{x}\| = 1\},$$

where $\mathbf{e}$ is the all ones vector. Note that $\mathbf{e}$ is an eigenvector of L associated with the zero eigenvalue. For simplicity, we often reference $W_r(L)$ as the restricted numerical range of a digraph.

Our definition of the restricted numerical range is motivated by its close connection to the algebraic connectivity for digraphs as defined in [22]. Indeed, let $\Gamma \in \mathbb{G}$ and let L be the Laplacian matrix of Γ. Then, the *algebraic connectivity* of Γ is defined by

$$\alpha(\Gamma) = \min_{\mathbf{x} \in \mathcal{S}} \mathbf{x}^T L\mathbf{x},$$

where

$$\mathcal{S} = \{\mathbf{x} \in \mathbb{R}^n \colon \ \mathbf{x} \perp \mathbf{e}, \ \ \|\mathbf{x}\| = 1\}.$$

Another related and useful quantity is

$$\beta(\Gamma) = \max_{\mathbf{x} \in \mathcal{S}} \mathbf{x}^T L\mathbf{x}.$$

We summarize this connection and other basic properties below. For the remainder of this section, let Q be a real $n \times (n-1)$ orthonormal matrix whose columns are orthogonal to $\mathbf{e}$.

Proposition 2.1. *Let $\Gamma \in \mathbb{G}$ and let L be the Laplacian matrix of Γ. Then, the following properties hold.*

(i) The restricted numerical range satisfies $W_r(L) = W(Q^T LQ)$.
(ii) The set $W_r(L)$ is invariant under re-ordering of the vertices of Γ.
(iii) The eigenvalues of L are contained in $W_r(L)$, except for the zero eigenvalue associated with the eigenvector $\mathbf{e}$.
(iv) The minimum real part of $W_r(L)$ is equal to $\alpha(\Gamma)$, and the maximum real part of $W_r(L)$ is equal to $\beta(\Gamma)$.

Proof.

(i) This result follows directly from the fact that Q acts as a length preserving bijection between $\mathbb{C}^{n-1}$ and the vectors in $\mathbb{C}^n$ that are orthogonal to $\mathbf{e}$.
(ii) The re-ordering of the vertices of Γ corresponds to a permutation matrix P such that $P^T LP$ is the Laplacian matrix of the re-ordered digraph. Furthermore, PQ is a real $n \times (n-1)$ orthonormal matrix with columns orthogonal to $\mathbf{e}$. Therefore,

$$W_r(L) = W(Q^T P^T LPQ) = W_r(P^T LP).$$

(iii) Since $\mathbf{e}$ is an eigenvector of L associated with the zero eigenvalue, we have

$$[Q\ \hat{\mathbf{e}}]^T L [Q\ \hat{\mathbf{e}}] = \begin{bmatrix} Q^T L Q & Q^T L \hat{\mathbf{e}} \\ \hat{\mathbf{e}}^T L Q & \hat{\mathbf{e}}^T L \hat{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} Q^T L Q & 0 \\ \hat{\mathbf{e}}^T L Q & 0 \end{bmatrix},$$

where $\hat{\mathbf{e}} = \mathbf{e}/\sqrt{n}$. Therefore, the eigenvalues of L are the union of the eigenvalues of $Q^T L Q$ and the zero eigenvalue associated with the eigenvector $\mathbf{e}$. The result follows from noting that $W_r(L)$ contains the eigenvalues of $Q^T L Q$.

(iv) By [13,23, Theorem 9], the minimum and maximum real parts of $W_r(L)$ is equal to the minimum and maximum eigenvalues, respectively, of the symmetric part of $Q^T L Q$. These values are attained in $W_r(L)$ by the associated eigenvectors of the symmetric part of $Q^T L Q$. The result follows since these eigenvectors can be taken to be real.

□

2.3. Examples

In Figure 1, the empty and complete digraph on 6 vertices are shown. It is clear that both the empty and complete digraphs are determined by their Laplacian spectrum, with $\sigma(L) = \{0^{(n)}\}$ and $\sigma(L) = \{0, n^{(n-1)}\}$, respectively, where the exponent denotes the algebraic multiplicity of the eigenvalue. We use these results to show that both the empty and complete digraphs are characterized by their restricted numerical range.

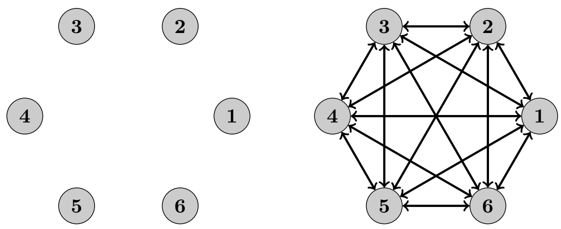

Figure 1. Empty and complete digraph on 6 vertices.

Theorem 2.2. *Let $\Gamma \in \mathbb{G}$ and let L be the Laplacian matrix of Γ. Then, Γ is an empty digraph if and only if $W_r(L) = \{0\}$.*

Proof. Suppose that Γ is an empty digraph. Then, $\mathbf{x}^* L \mathbf{x} = 0$ for all $\mathbf{x} \in \mathbb{C}^n$ and it follows that $W_r(L) = \{0\}$.

Conversely, suppose that $W_r(L) = \{0\}$. Then, it follows that $\sigma(L) = \{0^{(n)}\}$ and, therefore, Γ is an empty digraph. □

Theorem 2.3. *Let $\Gamma \in \mathbb{G}$ and let L be the Laplacian matrix of Γ. Then, Γ is a complete digraph if and only if $W_r(L) = \{n\}$.*

Proof. Suppose that Γ is a complete digraph. Then, L is a symmetric matrix with spectrum $\sigma(L) = \{0, n^{(n-1)}\}$. It follows that the eigenvectors of L associated with the eigenvalue n form an orthonormal basis for the subspace of vectors in $\mathbb{C}^n$ that are orthogonal to $\mathbf{e}$. Hence, $W_r(L) = \{n\}$.

Conversely, suppose that $W_r(L) = \{n\}$. Then, it follows that $\sigma(L) = \{0, n^{(n-1)}\}$ and, therefore, Γ is a complete digraph. □

Next, consider a directed cycle, as shown on 6 vertices in Figure 2. In what follows, we show that the directed cycle is characterized by its Laplacian spectrum and its restricted numerical range. Throughout, we denote the imaginary unit $\sqrt{-1}$ by i.

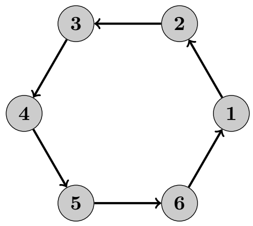

Figure 2. Directed cycle on 6 vertices.

Theorem 2.4. *Let $\Gamma \in \mathbb{G}$ and let L be the Laplacian matrix of Γ. Then, Γ is a directed cycle if and only if*

$$\sigma(L) = \left\{1 - e^{\mathrm{i}2\pi j/n} \colon \; j = 0, 1, \ldots, n-1\right\}. \tag{2}$$

Proof. Suppose that Γ is a directed cycle. Then, possibly after re-ordering the vertices, L is a circulant matrix whose first column vector is equal to $[1, 0, \ldots, 0, -1]^T$. As a circulant matrix with first column vector defined above, it is well-known that the eigenvalues of L satisfy (2), e.g., see [11, Section 2.2].

Conversely, suppose that the eigenvalues of L satisfy (2). In what follows, we show that $L = I - A$, where A is the adjacency matrix of Γ. Therefore,

$$\sigma(A) = \left\{e^{\mathrm{i}2\pi j/n} \colon \; j = 0, 1, \ldots, n-1\right\}.$$

and it follows from [3, Theorem 2.2.20] that A has index of cyclicity equal to n. Hence, Γ is a directed cycle.

For the sake of contradiction, suppose that $d^+(i) = 0$ for some $i \in V$. Then, L can be written in the Frobenius normal form (1) as

$$L = \begin{bmatrix} L_1 & L_{12} & \cdots & L_{1r} \\ & L_2 & \cdots & L_{2r} \\ & & \ddots & \vdots \\ & & & 0 \end{bmatrix},$$

where, for each $k = 1, \ldots, r-1$, L_k is an irreducible non-singular M-matrix. Hence, by [3, Theorem 6.2.3 (N38)], L_k^{-1} exists, is non-negative, and is irreducible. Also, by the Perron-Frobenius theorem [3, Theorem 2.1.4], the spectral radius $\rho(L_k^{-1})$ is a simple eigenvalue of L_k^{-1}. Now, let L_k denote the block with eigenvalue

$$\lambda = 1 - e^{\mathrm{i}2\pi/n},$$

which is the smallest, in magnitude, non-zero eigenvalue of L. Then, $\rho(L_k^{-1}) = |\lambda|^{-1}$ must be an eigenvalue of L_k^{-1}, which contradicts the eigenvalues of L satisfying (2).

Therefore, $d^+(i) \neq 0$ for all $i \in V$. Furthermore, since $\mathrm{tr}\,(L) = n$, we have $d^+(i) = 1$ for all $i \in V$. Hence, $L = I - A$, and the result follows. $\square$

In Lemma 2.5, we show that the normality of L implies the normality of Q^TLQ. Note that the converse of this result does not hold. Then, in Theorem 2.6, we prove that a directed cycle is characterized by its restricted numerical range.

Lemma 2.5. *Let L be the Laplacian matrix of any $\Gamma \in \mathbb{G}$. If L is normal, then Q^TLQ is normal.*

Proof. Suppose that L is a normal matrix. Then, L has an orthonormal eigenvector basis for $\mathbb{C}^n$. Since $\mathbf{e}$ is always an eigenvector of L, we denote this eigenvector basis by $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, where $\mathbf{v}_n = \mathbf{e}/\sqrt{n}$. It follows that for each $j = 1, \ldots, n-1$ there is a unique $\mathbf{x}_j \in \mathbb{C}^{n-1}$ such that $Q\mathbf{x}_j = \mathbf{v}_j$. Hence, $\{\mathbf{x}_1, \ldots, \mathbf{x}_{n-1}\}$ forms an orthonormal basis for $\mathbb{C}^{n-1}$, where each $\mathbf{x}_j$ is an eigenvector of Q^TLQ. Therefore, Q^TLQ is normal. □

Theorem 2.6. *Let $\Gamma \in \mathbb{G}$ and let L be the Laplacian matrix of Γ. Then, Γ is a directed cycle if and only if $W_r(L)$ is the complex polygon with vertices*

$$\left\{1 - e^{\mathrm{i}2\pi j/n} : \ j = 1, \ldots, n-1\right\}. \tag{3}$$

Proof. Suppose that Γ is a directed cycle on n vertices. Then, L is a circulant matrix with eigenvalues that satisfy (2). Furthermore, as a circulant matrix, it is well-known that the Fourier matrix provides a unitary diagonalization of L, e.g., see [11, Section 2.2]. Therefore, L is a normal matrix and, by Lemma 2.5, it follows that Q^TLQ is a normal matrix. As a normal matrix, it follows that $W(Q^TLQ)$ is equal to the convex hull of $\sigma(Q^TLQ)$ [13,23, Theorem 3], where $\sigma(Q^TLQ)$ is equal to the set in (3).

Conversely, suppose that $W(Q^TLQ)$ is the complex polygon with vertices in (3). By [13,23, Theorem 13], the vertices of this complex polygon are the eigenvalues of Q^TLQ. Therefore, $\sigma(L)$ satisfies (2), and it follows from Theorem 2.4 that Γ is a directed cycle. □

We conclude this section with a corollary that follows from Theorems 2.2, 2.3, 2.6 and Proposition 2.1 (iv.). We prove the converse of parts (i.) and (ii.) in Section 3.

Corollary 2.7. *Let $\Gamma \in \mathbb{G}$.*

(i) If Γ is an empty digraph, then

$$\alpha(\Gamma) = \beta(\Gamma) = 0.$$

(ii) If Γ is a complete digraph, then

$$\alpha(\Gamma) = \beta(\Gamma) = n.$$

(iii) If Γ is a directed cycle, then

$$\alpha(\Gamma) = 1 - \mathrm{Re}\left(e^{\mathrm{i}2\pi/n}\right)$$

and

$$\beta(\Gamma) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 1 - \mathrm{Re}\left(e^{\mathrm{i}\pi(n-1)/n}\right) & \text{if } n \text{ is odd.} \end{cases}$$

3. Digraphs with Singleton Restricted Numerical Range

In the previous section, we saw that the empty and complete digraphs have a singleton restricted numerical range. In this section, we characterize the class of digraphs that have a singleton restricted numerical range. Furthermore, we show that this class of digraphs can be completely described by a directed join.

Definition 3.1. The *directed join* of the digraphs $\Gamma = (V, E)$ and $\Gamma' = (V', E')$ is defined by

$$\Gamma \vec{\nabla} \Gamma' = \left(V \cup V', E \cup E' \cup \left\{(i, j) \colon\ i \in V, j \in V'\right\}\right).$$

Note that we allow Γ or Γ' to be the null digraph (zero vertices), in which case the directed join is a trivial operation that does not change the digraph of largest order.

We define a *k-imploding star* on n vertices, for $k \in \{0, 1, \ldots, n\}$, by the directed join $E_{n-k} \vec{\nabla} K_k$, where E_{n-k} is an empty digraph (zero edges) on $(n-k)$ vertices and K_k is a complete digraph on k vertices. Examples of 1-imploding and 2-imploding stars on 6 vertices are shown in Figure 3.

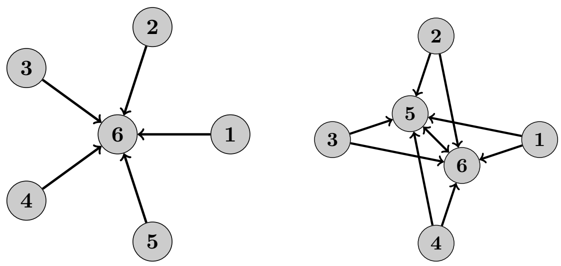

Figure 3. Examples of 1-imploding (left) and 2-imploding (right) stars on 6 vertices.

Theorem 3.2. *Let $\Gamma \in \mathbb{G}$ and let L be the Laplacian matrix of Γ. Then, Γ is a k-imploding star if and only if $W_r(L) = \{k\}$.*

Proof. Suppose that Γ is a k-imploding star. Then, possibly after re-ordering the vertices, L can be written in the form

$$L = kI - \mathbf{e} \left(\sum_{j=n-k+1}^{n} \mathbf{e}_j \right)^T,$$

where I is the identity matrix and $\mathbf{e}_j$ is the jth standard basis vector for $\mathbb{R}^n$. Let $\mathbf{x} \in \mathbb{C}^n$ be orthogonal to $\mathbf{e}$. Then,

$$\mathbf{x}^* L \mathbf{x} = \mathbf{x}^* \left(k\mathbf{x} - \mathbf{e} \left(\sum_{j=n-k+1}^{n} \mathbf{e}_j \right)^T \mathbf{x} \right) = k\mathbf{x}^*\mathbf{x}.$$

It immediately follows that $W_r(L) = \{k\}$.

Conversely, suppose that $W_r(L) = \{k\}$. In addition, let $\mathbf{x} = (\mathbf{e}_i - \mathbf{e}_j)/\sqrt{2}$, for some

$1 \leq i < j \leq n$. Then, since $\mathbf{x}$ is orthogonal to $\mathbf{e}$, we have

$$\begin{aligned} k &= \mathbf{x}^T L \mathbf{x} \\ &= \frac{1}{2}(\mathbf{e}_i - \mathbf{e}_j)^T (L\mathbf{e}_i - L\mathbf{e}_j) \\ &= \frac{1}{2}(l_{ii} + l_{jj} - l_{ij} - l_{ji}), \end{aligned}$$

where $L = [l_{ij}]_{i,j=1}^n$. Hence,

$$2k = l_{ii} + l_{jj} - l_{ij} - l_{ji} \tag{4}$$

for all $1 \leq i < j \leq n$. Also, since $\alpha(\Gamma) = \beta(\Gamma) = k$, it follows from [22, Lemma 8] that

$$d^+(i) = k - \frac{d^-(i)}{n-1},$$

for all vertices $i \in V$. Since $l_{ii} = d^+(i)$ must be an integer and k is an integer, we have $l_{ii} \in \{k-1, k\}$ for all $i = 1, \ldots, n$.

Now, let s be a non-negative integer such that $l_{ii} = k$, s times, and $l_{ii} = (k-1)$, $(n-s)$ times. Since $\sigma(L) = \left\{0, k^{(n-1)}\right\}$, the trace of L gives us

$$ks + (k-1)(n-s) = k(n-1),$$

from which it follows that $s = (n-k)$. Therefore, we can re-order the vertices such that $l_{ii} = k$, for $1 \leq i \leq (n-k)$, and $l_{ii} = (k-1)$, for $(n-k+1) \leq i \leq n$. Hence, after this re-ordering of vertices, we can write the Laplacian matrix in the form

$$L = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix},$$

where L_{11} is a $(n-k) \times (n-k)$ matrix whose diagonal entries are equal to k, and L_{22} is a $k \times k$ matrix whose diagonal entries are equal to $(k-1)$. If $(n-k+1) \leq i < j \leq n$, then (4) implies that

$$2k = 2(k-1) - l_{ij} - l_{ji},$$

from which it follows that $l_{ij} = l_{ji} = -1$. Hence, L_{22} is the Laplacian matrix of K_k and L_{21} is a zero matrix. Furthermore, applying (4) to vertices $1 \leq i < j \leq (n-k)$, gives us $L_{11} = kI$. Hence, every entry in L_{12} is equal to -1, and it follows that Γ is a k-imploding star. □

By Theorem 3.2, we know that every k-imploding star is in the class of digraphs with a singleton restricted numerical range. In fact, Theorem 3.3 implies that k-imploding stars are the only digraphs in this class. Before proving this result, note that, since L is real, $W_r(L)$ is symmetric with respect to the real axis and, hence, we only need to consider real singleton restricted numerical ranges. Also, note that the *complement* of the digraph $\Gamma = (V, E)$ is defined as the digraph $\overline{\Gamma} = (V, \overline{E})$, where

$$\overline{E} = \{(i,j) \in V \times V : \ (i,j) \notin E\}.$$

Theorem 3.3. *Let $\Gamma \in \mathbb{G}$ and let L be the Laplacian matrix of Γ. If $W_r(L) = \{k\}$ for some $k \in \mathbb{R}$, then k must be an integer between 0 and n (inclusive). Therefore, Γ is a k-imploding star.*

Proof. As a general M-matrix, the real part of every eigenvalue of L is non-negative [3, Theorem 6.4.6(E11)]. Since k must be an eigenvalue of L, it follows that $k \geq 0$.

For any $\Gamma \in \mathbb{G}$, by [22, Lemma 4], we have

$$\alpha(\Gamma) = n - \beta(\overline{\Gamma}).$$

Also, $\beta(\overline{\Gamma})$ must be non-negative; otherwise, the Laplacian spectrum of $\overline{\Gamma}$ would lie in the left-half of the complex plane, which would contradict [3, Theorem 6.4.6(E11)]. Therefore, $k = \alpha(\Gamma) \leq n$.

Now, suppose that $0 \leq k \leq n$ is not an integer. Since $\alpha(\Gamma) = \beta(\Gamma) = k$, it follows from [22, Lemma 8] that

$$d^+(i) = k - \frac{d^-(i)}{n-1},$$

for all vertices $i \in V$. Since the ith diagonal element of L is equal to $d^+(i)$, which must be an integer, the graph Laplacian can be written in the following form

$$L = \lfloor k \rfloor I - A,$$

where $A = [a_{ij}]_{i,j=1}^n$ is the adjacency matrix of Γ. Let $\mathbf{x} = (\mathbf{e}_i - \mathbf{e}_j)/\sqrt{2}$. Since $\mathbf{x}$ is orthogonal to $\mathbf{e}$, it follows that

$$\begin{aligned} k &= \lfloor k \rfloor - \mathbf{x}^T A \mathbf{x} \\ &= \lfloor k \rfloor - \left(\frac{a_{ii} + a_{jj} - a_{ij} - a_{ji}}{2} \right) \\ &= \lfloor k \rfloor - \left(\frac{a_{ij} + a_{ji}}{2} \right). \end{aligned}$$

Note that the above equation is a contradiction if either $a_{ij} = a_{ji} = 0$ or $a_{ij} = a_{ji} = 1$. Therefore, $a_{ij} = 1$ or $a_{ji} = 1$, but not both, for all $1 \leq i < j \leq n$; hence, Γ is a tournament digraph. Furthermore, since the sum of the entries of L must equal zero, we know that

$$\lfloor k \rfloor = \frac{n-1}{2}.$$

Hence, n must be an odd integer and

$$d^+(i) = d^-(i) = \frac{n-1}{2},$$

for all $i \in V$. Therefore, Γ is a regular tournament digraph for which it is well-known, e.g., see [5, Section 8], that its adjacency matrix, possibly after re-ordering the vertices, can be written as

$$A = P + P^2 + \cdots + P^{\lfloor k \rfloor},$$

where P is the permutation matrix corresponding to the permutation $(2, 3, \ldots, n, 1)$. Therefore, L is a circulant matrix whose eigenvalues satisfy

$$\sigma(L) = \left\{ \lfloor k \rfloor - \sum_{s=1}^{\lfloor k \rfloor} \left(e^{\mathrm{i}2\pi j/n} \right)^s : \; j = 0, 1, \ldots, n-1 \right\},$$

which contradicts $W_r(L)$ lying entirely on the real line. □

In the proof of Theorem 3.3, we saw that if $\alpha(\Gamma) = \beta(\Gamma)$ is not an integer, then Γ must be a regular tournament digraph. In particular, the following result clearly holds.

Corollary 3.4. *Let $\Gamma \in \mathbb{G}$ and let L be the Laplacian matrix of Γ. Then Γ is a regular tournament digraph if and only if n is odd and $W_r(L)$ is a vertical line segment with real part equal to $n/2$.*

Moreover, from the proofs of Theorems 3.2 and 3.3, we have the following result, which implies the converse of parts (i.) and (ii.) of Corollary 2.7.

Corollary 3.5. *Let $\Gamma \in \mathbb{G}$. Then, $\alpha(\Gamma) = \beta(\Gamma)$ if and only if Γ is a k-imploding star or a regular tournament digraph. In particular, Γ is a k-imploding star if and only if*

$$\alpha(\Gamma) = \beta(\Gamma) = k \in \{0, 1, \ldots, n\},$$

and Γ is a regular tournament digraph if and only if n is odd and

$$\alpha(\Gamma) = \beta(\Gamma) = \frac{n}{2}.$$

Finally, it is worth noting that for $1 \leq k \leq (n-2)$, a k-imploding star is not characterized by its Laplacian spectrum. Indeed, by simply removing an edge of the form $(i, j) \in V(E_{n-k}) \times V(K_k)$ and replacing it by an edge of the form $(i, j) \in V(E_{n-k}) \times V(E_{n-k})$, we obtain a cospectral digraph that is not isomorphic to $E_{n-k} \vec{\vee} K_k$. For example, consider the digraph in Figure 4, which is cospectral but not isomorphic to the 1-imploding star.

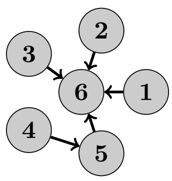

Figure 4. Digraph that is cospectral but not isomorphic to the 1-imploding star.

The above example raises the question: Are there digraphs that are characterized by their Laplacian spectrum, but are not characterized by their restricted numerical range? We suspect the answer to this question is no, which would make the restricted numerical range a more robust tool for characterizing digraphs. For instance, by the elliptical range theorem [14], the restricted numerical range of a digraph on 3 vertices is an ellipse with foci equal to the eigenvalues of the Laplacian not associated with $\mathbf{e}$. Thus, if two digraphs on 3 vertices are characterized by their Laplacian spectrum, then they must be characterized by their restricted numerical range.

4. Digraphs with Real Restricted Numerical Range

In the previous section, we characterized the class of digraphs that have a singleton restricted numerical range. In this section, we extend that characterization to include all digraphs with a real restricted numerical range. Note that every digraph in $\mathbb{G}$ on 1 or 2 vertices is a k-imploding star and, therefore, must have a real restricted numerical range. Hence, we may assume that we are working with digraphs with at least 3 vertices. In Theorem 4.2, we show that the digraphs with real restricted numerical range are balanced in the following sense.

Definition 4.1. A directed digraph $\Gamma \in \mathbb{G}$ with at least 3 vertices is called 3-*balanced* if for any three distinct vertices $i, j, k \in V$, we have

$$a_{ij} + a_{jk} + a_{ki} = a_{ik} + a_{kj} + a_{ji}, \tag{5}$$

where $A = [a_{ij}]_{i,j=1}^{n}$ is the adjacency matrix of Γ.

Theorem 4.2. *Let $\Gamma \in \mathbb{G}$ have at least 3 vertices and let L be the Laplacian matrix of Γ. Then, $W_r(L) \subset \mathbb{R}$ if and only if Γ is 3-balanced.*

Proof. Let $N = \{1, 2, \ldots, n\}$ and $k \in N$. Then, for all $i \in N \setminus \{k\}$, define $\mathbf{x}_i = \mathbf{e}_i - \mathbf{e}_k$. First, we show that $\mathbf{x}^* L\mathbf{x} \in \mathbb{R}$ for all $\mathbf{x} \in \mathbb{C}^n$ such that $\mathbf{x} \perp \mathbf{e}$ if and only if

$$\mathbf{x}_i^T L\mathbf{x}_j = \mathbf{x}_j^T L\mathbf{x}_i, \tag{6}$$

for all $i, j \in N \setminus \{k\}$. Indeed, note that if (6) fails for some $i, j \in N \setminus \{k\}$, then

$$(\mathbf{x}_i + \mathrm{i}\mathbf{x}_j)^* L (\mathbf{x}_i + \mathrm{i}\mathbf{x}_j) = \mathbf{x}_i^T L\mathbf{x}_i + \mathrm{i}\mathbf{x}_i^T L\mathbf{x}_j - \mathrm{i}\mathbf{x}_j^T L\mathbf{x}_i + \mathbf{x}_j^T L\mathbf{x}_j \notin \mathbb{R}.$$

To see that (6) is sufficient, note that $\{\mathbf{x}_i\}_{i \in N\setminus\{k\}}$ forms a basis for the subspace of vectors in $\mathbb{C}^n$ that are orthogonal to $\mathbf{e}$. Therefore, for each $\mathbf{x} \in \mathbb{C}^n$ such that $\mathbf{x} \perp \mathbf{e}$, we have

$$\mathbf{x} = \sum_{i \in N\setminus\{k\}} c_i \mathbf{x}_i,$$

where the c_i are complex scalars for $i \in N \setminus \{k\}$. Hence, if (6) holds, then

$$\begin{aligned}
\mathbf{x}^* L\mathbf{x} &= \sum_{i,j \in N\setminus\{k\}} \bar{c}_i c_j \mathbf{x}_i^T L\mathbf{x}_j \\
&= \sum_i |c_i|^2 \mathbf{x}_i^T L\mathbf{x}_i + \sum_{i<j} (\bar{c}_i c_j + c_i \bar{c}_j)\, \mathbf{x}_i^T L\mathbf{x}_j \in \mathbb{R},
\end{aligned}$$

where $\bar{c}$ denotes the complex conjugate of the complex scalar c.

With (6) established, the result follows from noting that, for each $i, j \in N \setminus \{k\}$, we have

$$\mathbf{x}_i^T L\mathbf{x}_j = l_{ij} - l_{ik} - l_{kj} + l_{kk} \quad \text{and} \quad \mathbf{x}_j^T L\mathbf{x}_i = l_{ji} - l_{jk} - l_{ki} + l_{kk}.$$

□

As an example, consider the digraphs in Figure 5. Note that both digraphs are 3-balanced. Hence, by Theorem 4.2, it follows that both digraphs have a real restricted numerical range. Moreover, note that both digraphs possess a structure very similar to the k-imploding stars shown in Figure 3. In particular, they are the directed join of two bidirectional digraphs.

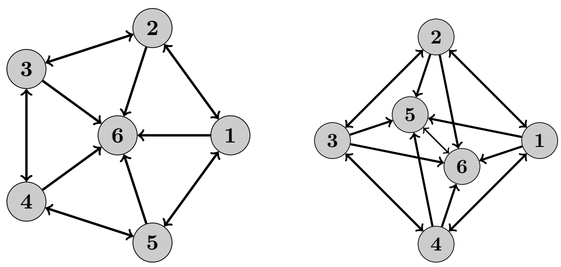

Figure 5. Digraphs with real restricted numerical range.

It turns out that all 3-balanced digraphs possess this structure. Before proving this result, we note that a digraph is 3-balanced if and only if its reversal is 3-balanced, where the reversal of a digraph is obtained by reversing the orientation of all the edges in the original digraph. Also, a digraph is said to be *bidirectional* if it is equal to its reversal.

Theorem 4.3. *Let* $\Gamma \in \mathbb{G}$ *have at least* 3 *vertices. Then,* Γ *is* 3*-balanced if and only if* Γ *is the directed join of two disjoint bidirectional digraphs in* $\mathbb{G}$*.*

Proof. Suppose that $\Gamma = (V, E)$ can be written as the directed join

$$\Gamma = S \vec{\vee} T,$$

where $S = (V_s, E_s)$ and $T = (V_t, E_t)$ are disjoint bidirectional digraphs. Let $i, j, k \in V$. Note that (5) holds if i, j, k all belong to either V_s or V_t. Therefore, since a graph is 3-balanced if and only if its reversal is 3 balanced, we may assume that $i, j \in V_s$ and $k \in V_t$. In this case, $a_{ij} = a_{ji}$, $a_{ik} = a_{jk}$, and $a_{ki} = a_{kj}$, which implies that (5) holds. Thus, Γ is a 3-balanced digraph.

Conversely, suppose that Γ is 3-balanced. In what follows, we use induction on the number of vertices to show that Γ must be the directed join of two disjoint bidirectional digraphs in $\mathbb{G}$. The base case, where Γ has exactly 3 vertices, is readily verified.

Suppose that the result holds for all 3-balanced digraphs on $n \geq 3$ vertices. Let Γ be a 3-balanced digraph on $(n+1)$ vertices. Furthermore, let Γ' be the subgraph of Γ formed from the vertex set $V' = \{1, 2, \ldots, n\}$. By the induction hypothesis, there exists disjoint bidirectional digraphs $S' = (V_{s'}, E_{s'})$ and $T' = (V_{t'}, E_{t'})$ such that

$$\Gamma' = S' \vec{\vee} T'.$$

Let $k = (n+1)$ and define

$$S = (V_{s'} \cup \{k\}, E \cap ((V_{s'} \cup \{k\}) \times (V_{s'} \cup \{k\})))$$

and

$$T = (V_{t'} \cup \{k\}, E \cap ((V_{t'} \cup \{k\}) \times (V_{t'} \cup \{k\}))) \,.$$

We split the induction step into two cases: First, when there is an edge from every vertex in $V_{s'}$ to k; second, where there is a vertex in $V_{s'}$ that does not connect to k. Note that if S' is null, then S contains only the vertex k. Furthermore, in order for Γ to be 3-balanced it follows that either $\Gamma = T$ or $\Gamma = S \vec{\vee} T'$. Similarly, if T' is null, then T contains only the vertex k and either $\Gamma = S$ or $\Gamma = S' \vec{\vee} T$. Therefore, during the induction step, we may assume that both S' and T' are not null.

Consider the first case, i.e., $(i, k) \in E$ for all $i \in V_{s'}$. If there is an $i \in V_{s'}$ such that both $(i, k) \in E$ and $(k, i) \in E$, i.e., the edge (i, k) is bidirectional, then every edge of the form $(j, k) \in E$, where $j \in V_{s'}$, must be bidirectional. Otherwise, there exists a bidirectional edge $(i, k) \in E$ and a non-bidirectional edge $(j, k) \in E$, where $i, j \in V_{s'}$, and it follows that the vertices i, j, k violate (5). Hence, if the edge (i, k) is bidirectional for some $i \in V_{s'}$, then S must be a bidirectional digraph. In this case, $(k, j) \in E$ and $(j, k) \notin E$ for all $j \in V_{t'}$. Indeed, if there exists a $j \in V_{t'}$ for which $(k, j) \notin E$ or $(j, k) \in E$, then for any $i \in V_{s'}$, the vertices i, j, k violate (5). Therefore, $\Gamma = S \vec{\vee} T'$, where S and T' are disjoint bidirectional digraphs.

Assume that each edge $(i, k) \in E$, where $i \in V_{s'}$ is not bidirectional. Then, T must be a bidirectional digraph. Otherwise, there exists a non-bidirectional edge $(j, k) \in E$ or $(k, j) \in E$, where $j \in V_{t'}$, and for any $i \in V_{s'}$ the vertices i, j, k violate (5). Therefore, $\Gamma = S' \vec{\vee} T$, where S' and T are disjoint bidirectional digraphs.

Consider the second case, i.e., there is an $i \in V_{s'}$ such that $(i, k) \notin E$. Then, $(k, j) \in E$ for all $j \in V_{t'}$. Otherwise, if there exists a $j \in V_{t'}$ such that $(k, j) \notin E$, then the vertices i, j, k violate (5). Furthermore, S must be a bidirectional digraph. Indeed, suppose for some $i \in V$ there exists an edge $(i, k) \in E$ which is not bidirectional. Then, for any $j \in V_{t'}$, the vertices i, j, k violate (5). Therefore, $\Gamma = S \vec{\vee} T'$, where S and T' are disjoint bidirectional digraphs. □

Corollary 4.4. *Let $\Gamma \in \mathbb{G}$ and let L be the Laplacian matrix of Γ. Then, $W_r(L) \subset \mathbb{R}$ if and only if Γ is the directed join of two disjoint bidirectional digraphs in $\mathbb{G}$.*

5. Conclusion

The restricted numerical range of the Laplacian matrix is a novel tool for characterizing digraphs and studying their algebraic connectivity. In Theorem 3.2 and Corollary 3.4, we showed that k-imploding stars and regular tournament digraphs are characterized by their restricted numerical range. Furthermore, in Corollary 3.5, we showed that both the k-imploding stars and regular tournament digraphs are determined by their algebraic connectivity, i.e., α and β values. Finally, in Theorem 4.2, we identified a new class of digraphs that are characterized by having a real restricted numerical range.

Note that the restricted numerical range has the practical advantage of avoiding the eigenvalue computation of the potentially defective Laplacian matrix. Instead, the boundary of the numerical range is approximated using complex polygons whose vertices are determined by the eigenvectors of a Hermitian matrix [12]. Moreover, the k-imploding stars offer an infinite number of digraphs that are characterized by their restricted numerical range, but are not characterized by their Laplacian spectrum.

Future research includes the relationship between digraphs that are characterized by their Laplacian spectrum and those that are characterized by their restricted numerical range. Also, we are interested in digraphs with a restricted numerical range that is a complex polygon and digraphs that have negative algebraic connectivity, i.e., their restricted numerical range intersects the left half of the complex plane.

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